Derivations of the finite-dimensional special odd Hamiltonian superalgebras

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Abstract The aim is to determine the derivations of the three series of finite-dimensional $\mathbb{Z}$-graded Lie superalgebras of Cartan-type over a field of characteristic $p > 3$, called the special odd Hamiltonian superalgebras. To that end we first determine the derivations of negative $\mathbb{Z}$-degree for the restricted and simple special odd Hamiltonian superalgebras by means of weight space decompositions. Then the results are used to determine the derivations of negative $\mathbb{Z}$-degree for the non-restricted and non-simple special odd Hamiltonian superalgebras. Finally the derivation algebras and the outer derivation algebras of those Lie superalgebras are completely determined.

Keywords special odd Hamiltonian superalgebra; restricted Lie superalgebra; derivation algebra

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0. Introduction

We work over a field $\mathbb{F}$ of positive characteristic. Using the divided powers algebras instead of the polynomial algebras one can construct eight families of finite dimensional $\mathbb{Z}$-graded Lie superalgebras of Cartan-type over $\mathbb{F}$, which are analogous to the vectorial Lie superalgebras over $\mathbb{C}$ (see [1, 3, 4, 15], for example). All these Lie superalgebras are subalgebras of the full (super)derivation algebras of the tensor products of the finite dimensional divided algebras and the exterior algebras, which are viewed as associative superalgebras in the obvious fashion. The

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derivation algebras were sufficiently studied for the modular Lie superalgebras of Cartan-type mentioned above (see [2, 6, 9, 10, 14, 16]), except the so-called special odd Hamiltonian superalgebras (see [5]).

The present paper aims to determine the derivation algebras of the special odd Hamiltonian superalgebras, especially, the outer derivation algebras. Our work is heavily depend on the results obtained in [5] and contains certain results obtained in 2005 in the thesis for master-degree by the third-named author [11]. We should mention that we use the method for Lie algebras [12] and benefit much from reading [12, 13].

1. Preliminaries

Hereafter $\mathbb{F}$ is a field of characteristic $p > 3$; $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ is the field of two elements. For a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we denote by $\mathrm{p}(a) = \theta$ the parity of a homogeneous element $a \in V_{\bar{\theta}}$, $\bar{\theta} \in \mathbb{Z}_2$. We assume throughout that the notation $\mathrm{p}(x)$ implies that x is a $\mathbb{Z}_2$-homogeneous element. $\mathbb{N}$ and $\mathbb{N}_0$ are the sets of positive integers and nonnegative integers, respectively. Let $m \geq 3$ denote a fixed positive integer and $\mathbb{N}^m$ the additive monoid of m-tuples of nonnegative integers. Fix two m-tuples $\underline{t} := (t_1, \ldots, t_m) \in \mathbb{N}^m$ and $\pi := (\pi_1, \ldots, \pi_m) \in \mathbb{N}^m$, where $\pi_i := p^{t_i} - 1$. Let $\mathcal{O}(m; \underline{t})$ be the divided power algebra with $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{A}(m; \underline{t})\}$, where $\mathbb{A}(m; \underline{t}) := \{\alpha \in \mathbb{N}_0^m \mid \alpha_i \leq \pi_i\}$. Write $|\alpha| := \Sigma_{i=1}^m \alpha_i$. For $\varepsilon_i := (\delta_{i1}, \delta_{i2}, \ldots, \delta_{im}) \in \mathbb{A}(m; \underline{t})$, we usually write x_i for $x^{(\varepsilon_i)}$, where $i = 1, \ldots, m$. Let $\Lambda(m)$ be the exterior superalgebra over $\mathbb{F}$ in m variables $x_{m+1}, x_{m+2}, \ldots, x_{2m}$. The tensor product $\mathcal{O}(m, m; \underline{t}) := \mathcal{O}(m; \underline{t}) \otimes_{\mathbb{F}} \Lambda(m)$ is an associative super-commutative superalgebra with a $\mathbb{Z}_2$-grading structure induced by the trivial $\mathbb{Z}_2$-grading of $\mathcal{O}(m; \underline{t})$ and the standard $\mathbb{Z}_2$-grading of $\Lambda(m)$. For $g \in \mathcal{O}(m, \underline{t})$, $f \in \Lambda(m)$, write gf for $g \otimes f$. Note that $x^{(\alpha)} x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}$ for $\alpha, \beta \in \mathbb{N}^m$, where $\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^m \binom{\alpha_i+\beta_i}{\alpha_i}$. Let

$$\mathbb{B}(m) := \left\{ \langle i_1, i_2, \ldots, i_k \rangle \mid m+1 \leq i_1 < i_2 < \cdots < i_k \leq 2m;\ k \in \overline{0, m} \right\}.$$

For $u := \langle i_1, i_2, \ldots, i_k \rangle \in \mathbb{B}(m)$, write $|u| := k$ and $x^u := x_{i_1} x_{i_2} \cdots x_{i_k}$. Notice that we also denote the index set $\{i_1, i_2, \ldots, i_k\}$ by u itself. For $u, v \in \mathbb{B}(m)$ with $u \cap v = \emptyset$, define $u + v$ to be the unique element $w \in \mathbb{B}(m)$ such that $w = u \cup v$. Similarly, if $v \subset u$, define $u - v$ to be the unique element $w \in \mathbb{B}(m)$ such that $w = u \setminus v$. Write $\omega = \langle m+1, \ldots, 2m \rangle$. Note that $\mathcal{O}(m, m; \underline{t})$ has a standard $\mathbb{F}$-basis $\{x^{(\alpha)} x^u \mid (\alpha, u) \in \mathbb{A} \times \mathbb{B}\}$. Put $\mathbf{Y_0} := \overline{1, m}$, $\mathbf{Y_1} := \overline{m+1, 2m}$ and $\mathbf{Y} := \overline{1, 2m}$. Let ∂_r be the superderivation of $\mathcal{O}(m, m; \underline{t})$ such that

$$\partial_r(x^{(\alpha)} x^u) := \begin{cases} x^{(\alpha - \varepsilon_r)} x^u, & r \in \mathbf{Y_0} \\ x^{(\alpha)} \partial x^u / \partial x_r, & i \in \mathbf{Y_1}. \end{cases}$$

The generalized Witt superalgebra $W(m, m; \underline{t})$ is spanned by all $f_r \partial_r$, where $f_r \in \mathcal{O}(m, m; \underline{t})$, $r \in \mathbf{Y}$. Note that $W(m, m; \underline{t})$ is a free $\mathcal{O}(m, m; \underline{t})$-module with basis $\{\partial_r \mid r \in \mathbf{Y}\}$. In particular, $W(m, m; \underline{t})$ has a so-called standard $\mathbb{F}$-basis $\{x^{(\alpha)} x^u \partial_r \mid (\alpha, u, r) \in \mathbb{A} \times \mathbb{B} \times \mathbf{Y}\}$. Note that $\mathcal{O}(m, m; \underline{t})$ possesses a so-called standard $\mathbb{Z}$-grading structure $\mathcal{O}(m, m; \underline{t}) = \bigoplus_{r=0}^{\xi} \mathcal{O}(m, m; \underline{t})_r$ by letting

$$\mathcal{O}(m, m; \underline{t})_r := \mathrm{span}_{\mathbb{F}} \{x^{(\alpha)} x^u \mid |\alpha| + |u| = r\}, \quad \xi := |\pi| + m = \sum_{i \in \mathbf{Y}_0} p^{t_i}.$$

This induces naturally a $\mathbb{Z}$-grading structure, also called standard,

$$W(m,m;\underline{t}) = \bigoplus_{i=-1}^{\xi-1} W(m,m;\underline{t})_i,$$

where

$$W(m,m;\underline{t})_i := \mathrm{span}_{\mathbb{F}}\{f\partial_r \mid r \in \mathbf{Y},\ f \in \mathcal{O}(m,m;\underline{t})_{i+1}\}.$$

Put

$$i' := \begin{cases} i+m, & \text{if } i \in \mathbf{Y_0} \\ i-m, & \text{if } i \in \mathbf{Y_1}, \end{cases} \qquad \mu(i) := \begin{cases} \bar{0}, & \text{if } i \in \mathbf{Y_0} \\ \bar{1}, & \text{if } i \in \mathbf{Y_1}. \end{cases}$$

Clearly, $\mathrm{p}(\partial_i) = \mu(i)$. Define the linear operator $\mathrm{T_H} : \mathcal{O}(m,m;\underline{t}) \longrightarrow W(m,m;\underline{t})$ such that

$$\mathrm{T_H}(a) := \sum_{i\in\mathbf{Y}} (-1)^{\mathrm{p}(\partial_i)\mathrm{p}(a)} \partial_i(a)\partial_{i'} \quad \text{for } a \in \mathcal{O}(m,m;\underline{t}).$$

Note that $\mathrm{T_H}$ is odd with respect to the $\mathbb{Z}_2$-grading and has degree -2 with respect to the $\mathbb{Z}$-grading. The following formula is well known:

$$[\mathrm{T_H}(a), \mathrm{T_H}(b)] = \mathrm{T_H}(\mathrm{T_H}(a)(b)) \quad \text{for}\, a, b \in \mathcal{O}(m,m;\underline{t})$$

and

$$HO(m,m;\underline{t}) := \{\mathrm{T_H}(a) \,|\, a \in \mathcal{O}(m,m;\underline{t})\}$$

is a finite-dimensional simple Lie superalgebra, called the odd Hamiltonian superalgebra [4, 8]. Put

$$\overline{HO}(m,m;\underline{t}) := \overline{HO}(m,m;\underline{t})_{\bar{0}} \oplus \overline{HO}(m,m;\underline{t})_{\bar{1}}$$

where for $\alpha \in \mathbb{Z}_2$,

$$\begin{aligned} \overline{HO}(m,m;\underline{t})_\alpha : \ = \ & \Big\{ \sum_{i\in\mathbf{Y}} a_i\partial_i \in W(m,m;\underline{t})_\alpha \,\Big| \\ & \partial_i(a_{j'}) = (-1)^{\mu(i)\mu(j)+(\mu(i)+\mu(j))(\alpha+\bar{1})}\partial_j(a_{i'}), i, j \in \mathbf{Y} \Big\}. \end{aligned}$$

We state certain basic results in [10, Proposition 1], which will be used in the following sections:

(i) Both $HO(m,m;\underline{t})$ and $\overline{HO}(m,m;\underline{t})$ are $\mathbb{Z}$-graded subalgebras of $W(m,m;\underline{t})$,

$$HO(m,m;\underline{t}) = \bigoplus_{i=-1}^{\xi-2} HO(m,m;\underline{t})_i; \quad \overline{HO}(m,m;\underline{t}) = \bigoplus_{i=-1}^{\xi-1} \overline{HO}(m,m;\underline{t})_i.$$

(ii) $HO(m,m;\underline{t})$ is a $\mathbb{Z}$-graded ideal of $\overline{HO}(m,m;\underline{t})$.

(iii) $\ker(\mathrm{T_H}) = \mathbb{F} \cdot 1$.

Let div : $W(m,m;\underline{t}) \longrightarrow \mathcal{O}(m,m;\underline{t})$ be the divergence, which is a linear mapping such that

$$\mathrm{div}(f_r\partial_r) = (-1)^{\mathrm{p}(\partial_r)\mathrm{p}(f_r)}\partial_r(f_r) \quad \text{for all}\, r \in \mathbf{Y}.$$

Note that div is an even $\mathbb{Z}$-homogeneous superderivation of $W(m,m;\underline{t})$ into the module $\mathcal{O}(m,m;\underline{t})$, that is

$$\mathrm{div}[E,D] = E(\mathrm{div}D) - (-1)^{\mathrm{p}(E)\mathrm{p}(D)}D(\mathrm{div}E) \quad \text{for all } D,E \in W(m,m;\underline{t}). \quad (1.1)$$

Putting

$$\begin{aligned} S'(m,m;\underline{t}) &:= \{D \in W(m,m;\underline{t}) \mid \mathrm{div}(D) = 0\}, \\ \overline{S}(m,m;\underline{t}) &:= \{D \in W(m,m;\underline{t}) \mid \mathrm{div}(D) \in \mathbb{F}\}, \end{aligned}$$

we have:

(i) Both $S'(m,m;\underline{t})$ and $\overline{S}(m,m;\underline{t})$ are $\mathbb{Z}$-graded subalgebras of $W(m,m;\underline{t})$:

$$S'(m,m;\underline{t}) = \bigoplus_{i=-1}^{\xi-1} S'(m,m;\underline{t})_i; \quad \overline{S}(m,m;\underline{t}) = \bigoplus_{i=-1}^{\xi-1} \overline{S}(m,m;\underline{t})_i.$$

(ii) $S'(m,m;\underline{t})$ is a $\mathbb{Z}$-graded ideal of $\overline{S}(m,m;\underline{t})$.

Here we write down the following symbols which will be frequently used in the future:

$$\begin{aligned} &\Delta := \sum_{i=1}^{m} \Delta_i, \quad \Delta_i := \partial_i\partial_{i'} \quad \text{for}\, i \in \mathbf{Y_0}; \\ &\nabla_i(x^{(\alpha)}x^u) := x^{(\alpha+\varepsilon_i)}x_{i'}x^u \quad \text{for}\, (\alpha,u) \in \mathbb{A}\times\mathbb{B}, i \in \mathbf{Y_0}; \\ &\Gamma_i^j := \nabla_j\Delta_i \quad \text{for } i,j \in \mathrm{Y}_0; \\ &\mathbf{I}(\alpha,u) := \{i \in \mathbf{Y_0} \mid \Delta_i(x^{(\alpha)}x^u) \neq 0\}; \\ &\widetilde{\mathbf{I}}(\alpha,u) := \{i \in \mathbf{Y_0} \mid \nabla_i(x^{(\alpha)}x^u) \neq 0\}; \\ &\mathcal{D}^* := \{x^{(\alpha)}x^u \mid \mathbf{I}(\alpha,u) \neq \emptyset, \widetilde{\mathbf{I}}(\alpha,u) \neq \emptyset\}; \\ &\mathcal{D}_1 := \{x^{(\alpha)}x^u \mid \mathbf{I}(\alpha,u) = \widetilde{\mathbf{I}}(\alpha,u) = \emptyset\}; \\ &\mathcal{D}_2 := \{x^{(\alpha)}x^u \mid \mathbf{I}(\alpha,u) = \emptyset,\, \widetilde{\mathbf{I}}(\alpha,u) \neq \emptyset\}. \end{aligned}$$

In this paper we mainly study the three series of Lie superalgebras:

$$\begin{aligned} &SHO(m,m;\underline{t}) := S'(m,m;\underline{t}) \cap HO(m,m;\underline{t}), \\ &SHO(m,m;\underline{t})^{(1)} := [SHO(m,m;\underline{t}), SHO(m,m;\underline{t})], \\ &SHO(m,m;\underline{t})^{(2)} := [SHO(m,m;\underline{t})^{(1)}, SHO(m,m;\underline{t})^{(1)}], \end{aligned}$$

called the special odd Hamiltonian superalgebras. By [5, Theorem 4.1] they are centerless and $SHO(m,m;\underline{t})^{(2)}$ is simple. Further informations for these Lie superalgebras can be found in [4, 5].

Convention 1.1. *For short, we usually omit the parameter* $(m,m;\underline{t})$ *and write* $\mathfrak{g}$ *for* SHO. *Sometime we also write* $\mathfrak{g}(\underline{t})$ *for* $\mathfrak{g}(m,m;\underline{t})$ *for* $t \in \mathbb{N}^m$.

We close this section by recalling the following general notion and basic facts. Suppose X is a finite dimensional $\mathbb{Z}$-graded Lie superalgebra, $X = \oplus_{i\in\mathbb{Z}} X_i$. By

$$\mathrm{Der}X := \mathrm{Der}_{\bar{0}}X \oplus \mathrm{Der}_{\bar{1}}X$$

denote the derivation algebra of X, which is also a $\mathbb{Z}$-graded Lie superalgebra,

$$\mathrm{Der}X = \sum_{i\in\mathbb{Z}} \mathrm{Der}_i X$$

where

$$\mathrm{Der}_i X = \{\phi \in \mathrm{Der}X \mid \phi(X_t) \subset X_{t+i},\ \forall t \in \mathbb{Z}\}.$$

As in the usual, write

$$\mathrm{Der}^{-}X := \mathrm{span}_{\mathbb{F}}\{\phi \in \mathrm{Der}_i X \mid i < 0\}, \quad \mathrm{Der}^{+}X := \mathrm{span}_{\mathbb{F}}\{\phi \in \mathrm{Der}_i X \mid i \geq 0\},$$

called the negative and nonnegative parts of the derivation algebra of X, respectively. The element in $\mathrm{Der}^{-}X$ is called negative degree derivation and the element in $\mathrm{Der}^{+}X$ is called nonnegative degree derivation.

2. Restrictedness and negative derivations

As mentioned in the introduction our main purpose is to determine the derivations of the special odd Hamiltonian superalgebras. Motivated by the method used in the modular Lie algebra theory [12, Lemma 6.1.3 and Theorem 7.1.2], in this paper we do not compute directly the derivations of the non-restricted and non-simple special odd Hamiltonian superalgebras but determine firstly the derivations (especially, those of negative degree) of the restricted and simple special odd Hamiltonian superalgebras. From [5] the Lie superalgebra $\mathfrak{g}^{(2)}$ is simple. Since we need the restrictednees of the Lie superalgebras under considerations in the process of determining derivations, in this section we first show that $\mathfrak{g}^{(2)}(\underline{t})$ is restricted if and only if $\underline{t} = \underline{1}$. Since a derivation is determined by its action on a generating set, we next give a generating set of the restricted Lie superalgebra $\mathfrak{g}^{(2)}(\underline{1})$. Finally, we determine the derivations of negative $\mathbb{Z}$-degree for $\mathfrak{g}^{(2)}(m, m; \underline{1})$, since it is enough for determining the derivations in the general case in the subsequent sections.

Let us introduce some symbols for later use:

$$\widetilde{\mathcal{G}} := \left\{ \mathrm{T_H}\left(x^{(\alpha)}x^{u} - \sum_{i\in\mathbf{I}(\alpha,u)} \Gamma_i^q(x^{(\alpha)}x^{u}) \right) \middle| x^{(\alpha)}x^{u} \in \mathcal{D}^*, q \in \widetilde{\mathbf{I}}(\alpha, u) \right\};$$

$$\mathfrak{A}_1 := \{\mathrm{T_H}(x^{(\alpha)}x^{u}) \mid \mathbf{I}(\alpha, u) = \widetilde{\mathbf{I}}(\alpha, u) = \emptyset\};$$

$$\mathfrak{A}_2 := \{\mathrm{T_H}(x^{(\alpha)}x^{u}) \mid \mathbf{I}(\alpha, u) = \emptyset, \widetilde{\mathbf{I}}(\alpha, u) \neq \emptyset\}.$$

We also write down some facts in [5]:

(i) [5, Lemma 2.2] For $f \in \mathcal{O}(m, m; \underline{t})$, $\mathrm{T_H}(f) \in \mathfrak{g}$ if and only if $\Delta(f) = 0$.

(ii) [5, Theorem 2.7] $\mathfrak{g}$ is spanned by $\widetilde{\mathcal{G}} \bigcup \mathfrak{A}_1 \bigcup \mathfrak{A}_2$ and $\mathfrak{g}$ is a $\mathbb{Z}$-graded subalgebra of $W(m, m; \underline{t})$, $\mathfrak{g} = \oplus_{i=-1}^{\xi-4}\mathfrak{g}_i$. Then $\mathfrak{g}$ is spanned by the elements of the form

$$\mathrm{T_H}\left(x^{(\alpha)}x^{u} - \sum_{i\in\mathbf{I}(\alpha,u)} \Gamma_i^q(x^{(\alpha)}x^{u}) \right) \quad q \in \widetilde{I}(\alpha, u). \tag{2.1}$$

For convenience, we call $x^{(\alpha)}x^u$ a leader of the element (2.1).

(iii) [5, Corollary 3.5] $\mathfrak{g}^{(1)}$ is spanned by $\widetilde{\mathcal{G}}\bigcup\mathfrak{A}_2$ and $\mathfrak{g}^{(1)}$ is a $\mathbb{Z}$-graded subalgebra of $W(m,m;\underline{t})$, $\mathfrak{g}^{(1)}=\oplus_{i=-1}^{\xi-4}(\mathfrak{g}^{(1)})_i$. Moreover,

$$(\mathfrak{g}^{(1)})_i=[(\mathfrak{g}^{(1)})_{-1},(\mathfrak{g}^{(1)})_{i+1}],\quad -1\leq i\leq \xi-5.$$

$$(\mathfrak{g}^{(1)})_{\xi-4}=\mathrm{span}_{\mathbb{F}}\left\{\mathrm{T_H}\left(x^{(\pi-\varepsilon_i)}x^{\omega-\langle i'\rangle}-\sum_{r\in\mathbf{Y_0}\setminus\{i\}}\Gamma_r^i\big(x^{(\pi-\varepsilon_i)}x^{\omega-\langle i'\rangle}\big)\right)\middle| i\in\mathbf{Y_0}\right\}.$$

Put $\mathcal{D}=\mathcal{D}_2\cup\mathcal{D}^*$. By (2.1), $\mathfrak{g}^{(1)}$ is spanned by the elements of the form

$$\mathrm{T_H}\left(x^{(\alpha)}x^u-\sum_{i\in\mathbf{I}(\alpha,u)}\Gamma_i^q(x^{(\alpha)}x^u)\right)\quad q\in\widetilde{I}(\alpha,u),\quad x^{(\alpha)}x^u\in\mathcal{D}. \tag{2.2}$$

(iv) [5, Theorem 3.8] $\mathfrak{g}^{(2)}$ is a $\mathbb{Z}$-graded subalgebras of $W(m,m;\underline{t})$, $\mathfrak{g}^{(2)}=\bigoplus_{i=-1}^{\xi-5}(\mathfrak{g}^{(2)})_i$. Moreover,

$$\begin{aligned}&(\mathfrak{g}^{(2)})_i=(\mathfrak{g}^{(1)})_i,\quad -1\leq i\leq\xi-5;\\&(\mathfrak{g}^{(2)})_{i-1}=[(\mathfrak{g}^{(2)})_{-1},(\mathfrak{g}^{(2)})_i],\quad 0\leq i\leq \xi-5.\end{aligned}$$

Theorem 2.1. *$\mathfrak{g}^{(2)}(\underline{t})$ is restricted if and only if $\underline{t}=\underline{1}$.*

Proof. Suppose $\underline{t}=\underline{1}$. Note that $W(m,m;\underline{1})$ is the full derivation algebra of the underlying algebra $\mathcal{O}(m,m;\underline{1})$. One sees that $W(m,m;\underline{1})$ is a restricted Lie superalgebra with respect to the usual p-power (mapping) and that the p-power fulfills that $(x_i\partial_i)^p=x_i\partial_i$ for $i\in\mathbf{Y}$ and vanishes on the other even standard basis elements, as in the Lie algebra case. Thus it is sufficient to show that the even part of $\mathfrak{g}^{(2)}(\underline{1})$ is closed under the p-power. Note that the even part of $\mathfrak{g}^{(2)}(1)$ is spanned by the elements of the form (2.2)

$$A:=\mathrm{T_H}\left(x^{(\alpha)}x^u-\sum_{i\in\mathbf{I}(\alpha,u)}\Gamma_i^q(x^{(\alpha)}x^u)\right)\quad q\in\widetilde{\mathbf{I}}(\alpha,u),\quad x^{(\alpha)}x^u\in\mathcal{D},$$

where $|u|$ is odd. It is sufficient to show that $A^p\in\mathfrak{g}^{(2)}(\underline{1})$. We shall frequently use the formula below without notice:

$$\mathrm{T_H}(x^\alpha x^u)^p=\begin{cases}\mathrm{T_H}(x^\alpha x^u), & \text{if } x^\alpha x^u=x_ix_{i'},\text{ for } i\in\mathbf{Y_0},\\ 0, & \text{otherwise,}\end{cases}$$

which is a direct consequence of [7, Proposition 5.1].

If $x^{(\alpha)}x^u=x_ix_{i'}$ for $i\in\mathbf{Y_0}$ then

$$A^p=\big(\mathrm{T_H}(x_ix_{i'}-\Gamma_i^qx_ix_{i'})\big)^p=\mathrm{T_H}(x_ix_{i'}-\Gamma_i^qx_ix_{i'})\in\mathfrak{g}^{(2)}(1),\ q\in\mathbf{Y_0},\ q\neq i.$$

Now assume that $x^{(\alpha)}x^u\neq x_ix_{i'}$ for $i\in\mathbf{Y_0}$ and let us show that $A^p=0$.

Case 1: $|u|>1$, that is, $|u|\geq 3$. Then $x^ux^{u-\langle i'\rangle-\langle j'\rangle}=x^{u-\langle i'\rangle}x^{u-\langle j'\rangle}=0$. It follows that $[\mathrm{T_H}(x^{(\alpha)}x^u),\mathrm{T_H}(\Gamma_r^qx^{(\alpha)}x^u)]=0$ and hence $A^p=0$.

Case 2: $|u| = 1$. Suppose $u = \{i'\}$, $i \in \mathbf{Y_0}$. Since A is a derivation of $\mathcal{O}(m, m; \underline{1})$, it suffices to show that $A^p(x_j) = 0$ for all $j \in \mathbf{Y}$. We consider the following two subcases:

Subcase 2.1: $j \in \mathbf{Y_0}$. We have

$$\begin{aligned} &A(x_j) = 0 \quad \text{for } j \in \mathbf{Y_0}\backslash\{i, q\};\\ &A^p(x_j) = ax^{(p\alpha-(p-1)\varepsilon_j)} = 0 \quad \text{for } j = i;\\ &A^p(x_j) = bx^{(p\alpha-p\varepsilon_j+\varepsilon_q)} = 0 \quad \text{for } j = q, \end{aligned}$$

where $a, b \in \mathbb{F}$.

Subcase 2.2: $j \in \mathbf{Y_1}$. We have

$$A^p(x_j) = cx^{(p\alpha-(p-1)\varepsilon_i-\varepsilon_{j'})}x_{i'} - dx^{(p\alpha-p\varepsilon_i-\varepsilon_{j'}+\varepsilon_q)}x_{q'}, \tag{2.3}$$

where $c, d \in \mathbb{F}$. In particular,

$$\Big(\mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{j'})}x_{i'}) - x^{(2\varepsilon_{j'})}x_j\Big)^3(x_j) = 0 \quad i \neq j'. \tag{2.4}$$

The equations (2.3) and (2.4) show $A^p(x_j) = 0$ unless $\alpha = \varepsilon_i + \varepsilon_{j'}$ with distinct i, j' and q. Note that

$$[\mathrm{T_H}(x^{(\varepsilon_i+\varepsilon_{j'})}x_{i'}), \mathrm{T_H}(x^{(\varepsilon_{j'}+\varepsilon_q)}x_{q'}] = 0 \text{ for distinct } i, j', q \in \mathbf{Y_0},$$

which implies that

$$\mathrm{T_H}\big(x^{(\varepsilon_i+\varepsilon_{j'})}x_{i'} - x^{(\varepsilon_{j'}+\varepsilon_q)}x_{q'}\big)^p = 0.$$

In conclusion, $A^p \in \mathfrak{g}^{(2)}(\underline{1})$ and hence $\mathfrak{g}^{(2)}(\underline{1})$ is a restricted Lie superalgebra.

Suppose conversely that $\mathfrak{g}^{(2)}(\underline{t})$ is a restricted Lie superalgebra. Then for every $i \in \mathbf{Y_0}$, $(\mathrm{ad}\partial_i)^p$ is an inner derivation and $(\mathrm{ad}\partial_i)^p$ is of $\mathbb{Z}$-degree ≥ -1. On the other hand we have $(\mathrm{ad}\partial_i)^p \in \mathrm{Der}_{-p}(\mathfrak{g}^{(2)}(\underline{1}))$. Consequently, $(\mathrm{ad}\partial_i)^p = 0$ for all $i \in \mathbf{Y_0}$ which forces $\underline{t} = \underline{1}$. The proof is complete. $\square$

The following lemma is simple but useful, the proof is similar to the one of the Lie algebra [13, Proposition 3.3.5].

Lemma 2.2. *Let $L = \oplus_{i=-r}^{s} L_i$ be a simple, finite dimensional, and $\mathbb{Z}$-graded Lie superalgebra. Then the following statements hold:*

(1) *L_{-r} and L_s are irreducible L_0-modules.*

(2) $[L_0, L_s] = L_s, \quad [L_0, L_{-r}] = L_{-r}.$

(3) $C_{L_{s-1}}(L_1) = 0, \quad [L_{s-1}, L_1] = L_s.$

(4) $C_L(L^+) = L_s, \quad C_L(L^-) = L_{-r}.$

Remark 2.3. *Let $T := \mathrm{span}_{\mathbb{F}}\{T_{ij} = \mathrm{T_H}(x_i x_{i'} - x_j x_{j'}) \mid i, j \in \mathbf{Y_0}, i \neq j\}$. Obviously, T is Abelian. From the proof of Theorem 2.1 we know $(T_{ij})^p = T_{ij}$ which shows T_{ij} is a toral. Consequently, T is a torus of $\mathfrak{g}$. In particular, T is a torus of the restricted Lie superalgebra of $\mathfrak{g}^{(2)}(\underline{1})$. A direct computation shows that*

$$[T_{ij}, \mathrm{T_H}\big(x^{(\alpha)}x^u\big)] = (\delta_{i'\in u} - \delta_{j'\in u} - \alpha_i + \alpha_j)\mathrm{T_H}\big(x^{(\alpha)}x^u\big). \tag{2.5}$$

Furthermore,

$$
\begin{aligned}
&[T_{ij}, \mathrm{T_H}\Big(x^{(\alpha)}x^u - \sum_{i\in\mathbf{I}(\alpha,u)} \Gamma_i^q(x^{(\alpha)}x^u)\Big)] \\
= \;& (\delta_{i'\in u} - \delta_{j'\in u} - \alpha_i + \alpha_j)\mathrm{T_H}\Big(x^{(\alpha)}x^u - \sum_{i\in\mathbf{I}(\alpha,u)} \Gamma_i^q(x^{(\alpha)}x^u)\Big),
\end{aligned}
$$

for $q \in \mathbf{Y_0}$.

Lemma 2.4. *Let* $L = \oplus_{i=-r}^{s} L_i$ *be a simple, finite-dimensional, and* $\mathbb{Z}$*-graded Lie superalgebra. Let* $M \subset L$ *be a subalgebra that contains* $L_{-1} \oplus L_1$*. If* $M \cap L_{s-1} \neq 0$*, then* $M = L$*.*

Proof. This is a direct consequence of Lemma 2.2. □

Lemma 2.5. $\mathfrak{g}^{(2)}(\underline{1})$ *is generated by* $\mathfrak{g}^{(2)}(\underline{1})_{-1} \oplus \mathfrak{g}^{(2)}(\underline{1})_1$*.*

Proof. Recall that $\mathfrak{g}^{(2)}(\underline{1}) = \mathfrak{g}^{(2)}(m, m; \underline{1})$ is a graded subalgebra of $W(\underline{1})$. Let M denote the subalgebra generated by $\mathfrak{g}^{(2)}(\underline{1})_{-1} \oplus \mathfrak{g}^{(2)}(\underline{1})_1$. We proceed by induction on m.

Suppose $m = 3$. Assume that $\mathfrak{g}^{(2)}(3, 3; \underline{1})_i \subset M$ for some $i \in \overline{1, 3p-6}$, and let A be an element of $\mathfrak{g}^{(2)}(3, 3; \underline{1})_{i+1}$ with a leader $x^{(\alpha)}x^u$ (cf (2.2)), that is

$$
A = \mathrm{T_H}\Big(x^{(\alpha)}x^u - \sum_{i\in\mathbf{I}(\alpha,u)} \Gamma_i^q(x^{(\alpha)}x^u)\Big) \quad q \in \widetilde{I}(\alpha, u), \quad x^{(\alpha)}x^u \in \mathcal{D}.
$$

Note that $\widetilde{\mathbf{I}}(\alpha, u) \neq \emptyset$. It is clear that $4 \leq |\alpha| + |u| \leq 3p - 3$. Let us show $A \in M$. We only have to consider the following cases:

Case 1: $|u| = 0$ and $|\alpha| \geq 4$. One may assume without loss of the generality that $\alpha_1 \geq 2$. Then

$$
\Big(\frac{\alpha_1(\alpha_1-1)}{2} - \alpha_1\alpha_2\Big)\mathrm{T_H}(x^{(\alpha)}) = \big[\mathrm{T_H}\big(x^{(\alpha-\varepsilon_1)}\big), \mathrm{T_H}\big(x^{(2\varepsilon_1)}x_{1'} - x^{(\varepsilon_1+\varepsilon_2)}x_{2'}\big)\big].
$$

(i) If $\alpha_2 = 0$, the equation shows that $\mathrm{T_H}(x^{(\alpha)}) \in M$.

(ii) If $\alpha_2 \neq 0$, we have

$$
\Big(\frac{\alpha_2(\alpha_2-1)}{2} - \alpha_1\alpha_2\Big)\mathrm{T_H}(x^{(\alpha)}) = \big[\mathrm{T_H}\big(x^{(\alpha-\varepsilon_2)}\big), \mathrm{T_H}\big(x^{(2\varepsilon_2)}x_{2'} - x^{(\varepsilon_1+\varepsilon_2)}x_{1'}\big)\big].
$$

If

$$
\frac{\alpha_1(\alpha_1-1)}{2} - \alpha_1\alpha_2 \equiv \frac{\alpha_2(\alpha_2-1)}{2} - \alpha_1\alpha_2 \equiv 0 \pmod p
$$

we obtain $\alpha_1 = \alpha_2 = p - 1$. Then $\alpha_3 < p - 1$. Since

$$
(1 + \alpha_3)\mathrm{T_H}(x^{(\alpha)}) = \big[\mathrm{T_H}\big(x^{(\alpha-\varepsilon_1)}\big), \mathrm{T_H}\big(x^{(2\varepsilon_1)}x_{1'} - x^{(\varepsilon_1+\varepsilon_3)}x_{3'}\big)\big],
$$

we have $\mathrm{T_H}(x^{(\alpha)}) \in M$.

Case 2: $|u| = 1$, $|\alpha| \geq 3$. One may assume without loss of generality that $u = \{1'\}$.

(i) Suppose $\alpha_1 = 0$ and $\alpha_2 \geq 2$. We have

$$\begin{aligned}&\left(\frac{\alpha_2(\alpha_2-1)}{2}+\alpha_2\right)\mathrm{T_H}(x^{(\alpha)}x_{1'})\\ =\ &\big[\mathrm{T_H}\big(x^{(\alpha-\varepsilon_2)}x_{1'}\big),\mathrm{T_H}\big(x^{(2\varepsilon_2)}x_{2'}-x^{(\varepsilon_1+\varepsilon_2)}x_{1'}\big)\big],\end{aligned}$$

and then $\mathrm{T_H}(x^{(\alpha)}x_{1'}) \in M$ if $\frac{\alpha_2(\alpha_2-1)}{2}+\alpha_2 \not\equiv 0 \pmod p$. On the other hand, if $\frac{\alpha_2(\alpha_2-1)}{2}+\alpha_2 \equiv 0 \pmod p$, we obtain $\alpha_2 = p-1$. Then $\alpha_3 < p-1$. Since

$$(1+\alpha_3)\mathrm{T_H}(x^{(\alpha)}x_{1'}) = \big[\mathrm{T_H}\big(x^{(\alpha-\varepsilon_2)}x_{1'}\big),\mathrm{T_H}\big(x^{(2\varepsilon_2)}x_{2'}-x^{(\varepsilon_2+\varepsilon_3)}x_{3'}\big)\big],$$

we have $A \in M$.

(ii) Suppose $\alpha_1 > 0$. One can assume that $\alpha_2 < p-1$. From Case 2 (i) we have, when $\alpha_3 < p-1$

$$\begin{aligned}&\mathrm{T_H}\big(x^{(\alpha)}x_{1'}-\Gamma_1^2x^{(\alpha)}x_{1'}\big)\\ =\ &-\big[\mathrm{T_H}\big(x^{(\alpha_1\varepsilon_1)}x_{2'}\big),\mathrm{T_H}\big(x^{((\alpha_2+1)\varepsilon_2+\alpha_3\varepsilon_3)}x_{1'}\big)\big] \in M.\end{aligned}$$

When $\alpha_3 = p-1$

$$\begin{aligned}&\mathrm{T_H}\big(x^{(\alpha)}x_{1'}-\Gamma_1^2x^{(\alpha)}x_{1'}\big)\\ =\ &\big[\mathrm{T_H}\big(x^{(\alpha_1\varepsilon_1+\varepsilon_3)}x_{2'}\big),\mathrm{T_H}\big(x^{((\alpha_2+1)\varepsilon_2+(\alpha_3-1)\varepsilon_3)}x_{1'}\big)\big] \in M.\end{aligned}$$

Similarly, we can obtain $\mathrm{T_H}(x^{(\alpha)}x_{1'}-\Gamma_1^3x^{(\alpha)}x_{1'}) \in M$, when $\alpha_3 < p-1$.

Case 3: $|u| = 2$, $|\alpha| \geq 2$. One can assume that $u = \{1', 2'\}$, $\alpha_3 < p-1$.

(i) Suppose $\alpha_1 = \alpha_2 = 0$. Applying case 2 (i) we have

$$\begin{aligned}&\mathrm{T_H}\big(x^{(\alpha)}x_{1'}x_{2'}\big)\\ =\ &-\frac{1}{\alpha_3+1}\big[\mathrm{T_H}\big(x^{(\alpha)}x_{1'}\big),\mathrm{T_H}\big(x^{(\varepsilon_3)}x_{2'}x_{3'}-\Gamma_3^1x^{(\varepsilon_3)}x_{2'}x_{3'}\big)\big] \in M.\end{aligned}$$

(ii) Suppose $\alpha_1 > 0$, $\alpha_1 < p-1$ or $\alpha_2 < p-1$. From Case 2 (ii) and Case 3 (i) we have

$$\begin{aligned}&(\alpha_3+1)\mathrm{T_H}\left(x^{(\alpha)}x_{1'}x_{2'}-\sum_{r\in\mathbf{I}(\alpha,\{1',2'\})}\Gamma_r^3x^{(\alpha)}x_{1'}x_{2'}\right)\\ =\ &\big[\mathrm{T_H}\big(x^{(\alpha_3\varepsilon_3)}x_{1'}x_{2'}\big),\mathrm{T_H}\big(x^{(\alpha-(\alpha_3-1)\varepsilon_3)}x_{3'}-\Gamma_3^qx^{(\alpha-(\alpha_3-1)\varepsilon_3)}x_{3'}\big)\big] \in M,\end{aligned}$$

where $q = 1$ or 2 such that $\alpha_q < p-1$.

(iii) Suppose $\alpha_1 = \alpha_2 = p-1$, $\alpha_3 < p-2$. Applying Cases 2 and 3 (ii), we obtain

$$\begin{aligned}&\mathrm{T_H}\left(x^{(\alpha)}x_{1'}x_{2'}-\sum_{r\in\{1,2\}}\Gamma_r^3x^{(\alpha)}x_{1'}x_{2'}\right)\\ =\ &-\big[\mathrm{T_H}\big(x^{(p-2)\varepsilon_1}x_{2'}x_{3'}\big),\mathrm{T_H}\big(x^{(\varepsilon_1+(p-1)\varepsilon_2+(\alpha_3+1)\varepsilon_3)}x_{1'}-x^{((p-1)\varepsilon_2+(\alpha_3+2)\varepsilon_3)}x_{3'}\big)\big]\\ &\in M.\end{aligned}$$

Now suppose $m > 3$. Let

$$\overline{L} := \mathrm{span}_{\mathbb{F}}\{\overline{X}\},$$

where

$$\overline{X} := \left\{ \mathrm{T_H}\big(x^{(\alpha)}x^u - \sum_{r\in\mathbf{I}(\alpha,u)} \Gamma_r^q x^{(\alpha)}x^u\big) \;\middle|\; \begin{array}{l} \alpha = \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_3\varepsilon_3 \\ u \subset \{1', 2', 3', \} \\ |\alpha| + |u| \leq 3p - 3 \\ x^{(\alpha)}x^u \in \mathcal{D}, \quad q \in \widetilde{I}(\alpha, u) \end{array} \right\} \subset \mathfrak{g}^{(2)}(\underline{1}).$$

Let

$$L' := \mathrm{span}_{\mathbb{F}}\{X'\},$$

where

$$X' := \left\{ \mathrm{T_H}\big(x^{(\alpha)}x^u - \sum_{r\in\mathbf{I}(\alpha,u)} \Gamma_r^q x^{(\alpha)}x^u\big) \;\middle|\; \begin{array}{l} \alpha_i = 0,\ i \in \{1, 2, 3\} \\ u \subset \omega' := \omega - \{1', 2', 3'\} \\ |\alpha| + |u| \leq (m-3)p - 3 \\ x^{(\alpha)}x^u \in \mathcal{D}, \quad q \in \widetilde{I}(\alpha, u) \end{array} \right\} \subset \mathfrak{g}^{(2)}(\underline{1}).$$

Obviously, $\overline{L} \cong \mathfrak{g}^{(2)}(3, 3; \underline{1})$ and $L' \cong \mathfrak{g}^{(2)}(m-3, m-3; \underline{1})$.

Let $\overline{\mathcal{E}} := (p-1)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, $\mathcal{E}' := (p-1)(\varepsilon_4 + \cdots + \varepsilon_m)$, $\mathcal{E} := \overline{\mathcal{E}} + \mathcal{E}'$. The induction hypothesis are applied to these algebras yield:

$$\begin{aligned} \overline{\Omega} &:= \mathrm{T_H}\big(x^{(\overline{\mathcal{E}}-\varepsilon_1)}x_{3'} - x^{(\overline{\mathcal{E}}-\varepsilon_3)}x_{1'}\big) \in \overline{L} \subset M, \\ \Omega' &:= \mathrm{T_H}\Bigg(x^{(\mathcal{E}'-\varepsilon_{m-1}-\varepsilon_m)}x^{\omega'-\langle m-1'\rangle} \\ &\qquad - \sum_{r\in\mathcal{A}} \Gamma_r^{m-1} x^{(\mathcal{E}'-\varepsilon_{m-1}-\varepsilon_m)}x^{\omega'-\langle m-1'\rangle}\Bigg) \in L' \subset M, \end{aligned}$$

where $\mathcal{A} = \mathbf{I}(\mathcal{E}' - \varepsilon_{m-1} - \varepsilon_m, \omega' - \langle m-1'\rangle)$. Noting that

$$\begin{aligned} \Omega &:= \mathrm{T_H}\big(x^{(\varepsilon_1+\varepsilon_2+\varepsilon_{m-1})}x_{2'} - x^{(\varepsilon_1+\varepsilon_{m-1}+\varepsilon_m)}x_{m'}\big) \\ &= \big[\mathrm{T_H}\big(x^{(\varepsilon_{m-1}+\varepsilon_m)}x_{2'}\big), \mathrm{T_H}\big(x^{(\varepsilon_1+\varepsilon_2)}x_{m'}\big)\big] \in M \end{aligned}$$

and

$$B := [\Omega, \overline{\Omega}] = -\mathrm{T_H}\big(x^{(\overline{\mathcal{E}}+\varepsilon_{m-1})}x_{3'} - x^{(\overline{\mathcal{E}}-\varepsilon_3+\varepsilon_{m-1}+\varepsilon_m)}x_{m'}\big) \in M,$$

we have

$$\begin{aligned} C := [\Omega', B] &= \mathrm{T_H}\Bigg(x^{(\mathcal{E}-\varepsilon_3-\varepsilon_m)}x^{\omega'-\langle m-1'\rangle} \\ &\qquad - \sum_{r\in\mathcal{A}} \Gamma_r^3 x^{(\mathcal{E}-\varepsilon_3-\varepsilon_m)}x^{\omega'-\langle m-1'\rangle}\Bigg) \\ &\in M \cap \big(g^{(2)}(\underline{1})\big)_{mp-8}. \end{aligned}$$

Note that

$$\mathbf{I}(\mathcal{E} - \varepsilon_3 - \varepsilon_m, \omega' - \langle m-1'\rangle) = \mathbf{I}(\mathcal{E}' - \varepsilon_{m-1} - \varepsilon_m, \omega - \langle m-1'\rangle).$$

Putting

$$\begin{aligned} D &:= \mathrm{T_H}\big(x^{(\varepsilon_m)}x_{1'}x_{2'}x_{m-1'}\big)\\ &= -\frac{1}{2}\big[\mathrm{T_H}\big(x^{(\varepsilon_m)}x_{1'}x_{2'}\big), \mathrm{T_H}\big(x^{(\varepsilon_m)}x_{m-1'}x_{m'}+x^{(\varepsilon_1)}x_{1'}x_{m-1'}\big)\big] \in M, \end{aligned}$$

we have

$$\begin{aligned} [D,C] &= \mathrm{T_H}\Big(x^{(\mathcal{E}-\varepsilon_3-\varepsilon_m)}x^{\omega-\langle 3'\rangle-\langle m'\rangle}\\ &\qquad -\sum_{r'\in\omega\backslash\{3',m'\}}\Gamma_r^3 x^{(\mathcal{E}-\varepsilon_3-\varepsilon_m)}x^{\omega-\langle 3'\rangle-\langle m'\rangle}\Big)\\ &+ (-1)^{m-5}\mathrm{T_H}\Big(x^{(\mathcal{E}-\varepsilon_1-\varepsilon_3)}x^{\omega-\langle 1'\rangle-\langle 3'\rangle}\\ &\qquad -\sum_{r'\in\omega\backslash\{1',3'\}}\Gamma_r^3 x^{(\mathcal{E}-\varepsilon_1-\varepsilon_3)}x^{\omega-\langle 1'\rangle-\langle 3'\rangle}\Big)\\ &- (-1)^{m-5}\mathrm{T_H}\Big(x^{(\mathcal{E}-\varepsilon_2-\varepsilon_3)}x^{\omega-\langle 2'\rangle-\langle 3'\rangle}\\ &\qquad -\sum_{r'\in\omega\backslash\{2',3'\}}\Gamma_r^3 x^{(\mathcal{E}-\varepsilon_2-\varepsilon_3)}x^{\omega-\langle 2'\rangle-\langle 3'\rangle}\Big)\\ &+ \mathrm{T_H}\Big(x^{(\mathcal{E}-\varepsilon_3-\varepsilon_{m-1})}x^{\omega-\langle 3'\rangle-\langle m-1'\rangle}\\ &\qquad -\sum_{r'\in\omega\backslash\{3',m-1'\}}\Gamma_r^3 x^{(\mathcal{E}-\varepsilon_3-\varepsilon_{m-1})}x^{\omega-\langle 3'\rangle-\langle m-1'\rangle}\Big)\\ &\in M\cap\big(g^{(2)}(\underline{1})\big)_{mp-6}. \end{aligned}$$

Applying Lemma 2.4, we have $\mathfrak{g}^{(2)}(\underline{1}) = M$, which is generated by $\mathfrak{g}^{(2)}(\underline{1})_{-1}\oplus\mathfrak{g}^{(2)}(\underline{1})_1$. □

Lemma 2.6. *Let $L=\oplus_{i=-r}^{s}L_i$ be a $\mathbb{Z}$-graded and centerless Lie superalgebra and $T\subset L_0\cap L_{\bar 0}$ be an Abelian subalgebra of L such that $\mathrm{ad}x$ is semisimple for all $x\in T$. If $\varphi\in\mathrm{Der}_{\mathbb{F}}(L)$ is homogeneous of degree t, there is $e\in L_t$ such that $\big(\varphi-\mathrm{ad}e\big)\big|_T=0$.*

Proof. The proof is similar to the one of [13, Proposition 8.4]. □

Convention 2.7. *Hereafter we suppose $m>3$ for simplicity.*

Theorem 2.8. $\mathrm{Der}^-(\mathfrak{g}^{(2)}(\underline{1}))=\sum_{i\in\mathbf{Y}}\mathbb{F}\mathrm{ad}(\partial_i)$.

Proof. Let φ be a homogeneous derivation of degree $t<0$. From Lemma 2.6 we may assume that $\varphi(\mathfrak{g}^{(2)}(\underline{1})_{-1}+T)=0$, where

$$T=\mathrm{span}_{\mathbb{F}}\{T_{ij}=\mathrm{T_H}(x_ix_{i'}-x_jx_{j'})\mid i,j\in\mathbf{Y_0}, i\neq j\}$$

is a torus of $\mathfrak{g}^{(2)}(\underline{1})$ (see Remark 2.3). Since $\mathfrak{g}^{(2)}(\underline{1})$ is generated by $\mathfrak{g}^{(2)}(\underline{1})_{-1}\oplus\mathfrak{g}^{(2)}(\underline{1})_1$, we may assume that $t\in\{-1,-2\}$, and only have to show $\varphi(\mathfrak{g}^{(2)}(\underline{1})_1)=0$.

Case 1: $t=-2$. We can assert that $\varphi(\mathrm{T_H}(x^{(3\varepsilon_i)}))=0$, for any $i\in\mathbf{Y_0}$.

Assume that
$$\varphi(\mathrm{T_H}(x^{(3\varepsilon_i)}))=\sum_{r\in\mathbf{Y}}a_r\partial_r.$$
Applying φ to the equation
$$[\mathrm{T_H}(x^{(3\varepsilon_i)}),\mathrm{T_H}(x^{(\varepsilon_k)}x_{j'})]=0$$
where $i,j,k\in\mathbf{Y_0}$ are distinct, we can obtain $a_{j'}=a_k=0$, then
$$\varphi(\mathrm{T_H}(x^{(3\varepsilon_i)}))=a_i\partial_i+a_j\partial_j+a_{i'}\partial_{i'}+a_{k'}\partial_{k'}.$$
Applying φ to the equation
$$[T_{ij},\mathrm{T_H}(x^{(3\varepsilon_i)})]=-3\mathrm{T_H}(x^{(3\varepsilon_i)}),$$
we obtain $a_i=a_j=a_{i'}=a_{k'}=0$. Hence $\varphi(\mathrm{T_H}x^{(3\varepsilon_i)})=0$.

From a direct and simple computation we can obtain that $\mathfrak{g}^{(2)}(\underline{1})_1$ is generated by
$$\mathfrak{g}^{(2)}(\underline{1})_0\oplus\sum_{i\in\mathbf{Y_0}}\mathbb{F}\mathrm{T_H}(x^{(3\varepsilon_i)}).$$
Hence $\varphi=0$.

Case 2: $t=-1$. We can assert that $\varphi(\mathrm{T_H}(x^{(2\varepsilon_i)}))=0$ and $\varphi(\mathrm{T_H}(x_{i'}x_{j'}))=0$ $i,j\in\mathbf{Y_0}$, $i\neq j$.

Assume that
$$\varphi(\mathrm{T_H}(x^{(2\varepsilon_i)}))=\sum_{r\in\mathbf{Y}}b_r\partial_r.$$
Applying φ to the equation
$$[\mathrm{T_H}(x^{(2\varepsilon_i)}),T_{jk}]=0,$$
where $i,j,k\in\mathbf{Y_0}$ are distinct, we can obtain $b_j=b_{j'}=b_k=b_{k'}=0$, then
$$\varphi(\mathrm{T_H}(x^{(2\varepsilon_i)}))=b_i\partial_i+b_{i'}\partial_{i'}.$$
Applying φ to the equation
$$[T_{ik},\mathrm{T_H}(x^{(2\varepsilon_i)})]=-2\mathrm{T_H}(x^{(2\varepsilon_i)}),$$
we obtain $b_i=b_{i'}=0$. Hence $\varphi(\mathrm{T_H}x^{(2\varepsilon_i)})=0$.

Assume that
$$\varphi(\mathrm{T_H}(x_{i'}x_{j'}))=\sum_{r\in\mathbf{Y}}c_r\partial_r.$$
Applying φ to the equation
$$[\mathrm{T_H}(x_{i'}x_{j'}),T_{ij}]=0,$$
we can obtain $c_j=c_{j'}=c_i=c_{i'}=0$, hence
$$\varphi(\mathrm{T_H}(x_{i'}x_{j'}))=\sum_{r\in\mathbf{Y}\setminus\{i,i',j,j'\}}c_r\partial_r.$$

For $m \geq 4$, we can put $k, l \in \mathbf{Y_0}$ satisfying $k \neq l$, $k, l \neq i, j$. Applying φ to the equation

$$[T_{kl}, \mathrm{T_H}(x_{i'}x_{j'})] = 0,$$

we obtain $b_i = b_{i'} = 0$. Hence $\varphi(\mathrm{T_H}(x_{i'}x_{j'})) = 0$. Note that $\mathfrak{g}^{(2)}(\underline{1})_0$ is generated by

$$T \oplus \sum_{i,j\in\mathbf{Y_0}, i\neq j} \mathbb{F}\mathrm{T_H}(x^{(2\varepsilon_i)}) \oplus \sum_{i,j\in\mathbf{Y_0}, i\neq j} \mathbb{F}\mathrm{T_H}(x_{i'}x_{j'}).$$

Hence $\varphi(\mathfrak{g}^{(2)}(\underline{1})_0) = 0$. Consequently,

$$[\mathfrak{g}^{(2)}(\underline{1})_{-1}, \varphi\big(\mathfrak{g}^{(2)}(\underline{1})_1\big)] = \varphi\big([\mathfrak{g}^{(2)}(\underline{1})_{-1}, \mathfrak{g}^{(2)}(\underline{1})_1]\big) = 0.$$

By means of the transitiveness of the simple algebra, we have $\varphi\big(\mathfrak{g}^{(2)}(\underline{1})_1\big) = 0$. Hence $\varphi = 0$. From Lemma 2.6 the conclusion holds. □

3. Derivations

In this section, we will determine derivations of $\mathfrak{g}$, $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$. Firstly, we study the derivations of negative $\mathbb{Z}$-degree for $\mathfrak{g}$, $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$, by virtue of the same subjects of the restricted Lie superalgebra $\mathfrak{g}^{(1)}(\underline{1})$. Secondly, we discuss the normalizers of $\mathfrak{g}$, $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ in W. Finally, we obtain the derivations of $\mathfrak{g}$, $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$.

Lemma 3.1. *Let M denote a subalgebra of $\mathfrak{g}(\underline{s})$, $\underline{1} \leq \underline{k} \leq \underline{s}$. If*

$$\mathfrak{g}^{(2)}(\underline{k}) + \mathbb{F}\mathrm{T_H}\big(x^{(p^{k_i}+1)\varepsilon_i}\big) \subset M$$

for some i, then $\mathfrak{g}^{(2)}(\underline{k} + \varepsilon_i) \subset M$.

Proof. Observing that under consideration $\mathfrak{g}^{(2)}(\underline{k} + \varepsilon_i)$ is a simple $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}(\underline{s})$, that is

$$g^{(2)}(\underline{k} + \varepsilon_i) = \bigoplus_{r=-1}^{n} (g^{(2)}(\underline{k} + \varepsilon_i))_r,$$

where $n = \Sigma_{j=1}^m p^{k_j} - 4$. From Lemma 2.2 we only have to prove that

$$M \cap \mathfrak{g}^{(2)}(\underline{k} + \varepsilon_i)_{n-1} \neq (0).$$

In order to accomplish this, we observe that:
for $1 \leq a$, $1 \leq b \leq p-1$, $1 \leq r \leq p$

$$\binom{rp^a - b}{p^a} \equiv \binom{r-1}{1} = r - 1 \pmod{p}$$

holds. Assume inductively and without loss of the generality that M contains

$$\begin{aligned} E_r \quad := \quad \mathrm{T_H}\Bigg(& x^{(\mathcal{E}_k - \varepsilon_1 - \varepsilon_2 + (r-1)p^{k_i}\varepsilon_i)} x^{\omega - \langle 1' \rangle} \\ & - \sum_{q\in\mathcal{B}_1} \Gamma_q^1 x^{(\mathcal{E} - \varepsilon_1 - \varepsilon_2 + (r-1)p^{k_i}\varepsilon_i)} x^{\omega - \langle 1' \rangle}\Bigg), \end{aligned}$$

where $1 \le r \le p-1$, $i \ne 1, 2$ and

$$\begin{aligned} \mathcal{E}_k &= (p^{k_1}-1)\varepsilon_1 + (p^{k_2}-1)\varepsilon_2 + \cdots + (p^{k_m}-1)\varepsilon_m; \\ \mathcal{B}_1 &= \mathbf{I}(\mathcal{E}_k - \varepsilon_1 - \varepsilon_2 + (r-1)p^{k_i}\varepsilon_i, \omega - \langle 1' \rangle). \end{aligned}$$

Then, we obtain

$$\begin{aligned} & [\mathrm{T_H}(x^{(p^{k_i}+1)\varepsilon_i}), E_r] \\ = \; & r(-1)^{i-2}\mathrm{T_H}\Big(x^{(\mathcal{E}_k - \varepsilon_1 - \varepsilon_2 + rp^{k_i}\varepsilon_i)} x^{\omega - \langle 1' \rangle - \langle i' \rangle} \\ & - \sum_{q \in \mathcal{B}_2} \Gamma_q^1 x^{(\mathcal{E}_k - \varepsilon_1 - \varepsilon_2 + rp^{k_i}\varepsilon_i)} x^{\omega - \langle 1' \rangle - \langle i' \rangle} \Big), \end{aligned}$$

where $\mathcal{B}_2 = \mathbf{I}(\mathcal{E}_k - \varepsilon_1 - \varepsilon_2 + rp^{k_i}\varepsilon_i, \omega - \langle 1' \rangle - \langle i' \rangle)$.

By induction we obtain

$$\begin{aligned} & \mathrm{T_H}\Big(x^{(\mathcal{E}_k - \varepsilon_1 - \varepsilon_2 + (p-1)p^{k_i}\varepsilon_i)} x^{\omega - \langle 1' \rangle - \langle i' \rangle} \\ & - \sum_{q \in \mathcal{B}_2} \Gamma_q^1 x^{(\mathcal{E}_k - \varepsilon_1 - \varepsilon_2 + (p-1)p^{k_i}\varepsilon_i)} x^{\omega - \langle 1' \rangle - \langle i' \rangle} \Big) \\ & \in M \cap \mathfrak{g}^{(2)}(\underline{k} + \varepsilon_i)_{n-1}. \end{aligned}$$

Hence the assertion holds. □

Theorem 3.2. *Let X be a $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}(m, m; \underline{t})$ containing $\mathfrak{g}^{(\infty)}$ and $\underline{s}$ be any element of $\mathbb{N}^m$ with $\underline{t} \le \underline{s}$. Then*

$$\mathrm{Der}^-(X, \mathfrak{g}(\underline{s})) = \mathrm{span}_{\mathbb{F}}\Big\{ \{(\mathrm{ad}_X(\partial_i))^{p^{k_i}} \mid i \in \mathbf{Y_0},\, 1 \le k_i < t_i\} \cup \{\mathrm{ad}_X(\partial_i) \mid i \in \mathbf{Y}\} \Big\}.$$

Proof. Let T be the torus of X_0 mentioned in Remark 2.3. Then

$$\mathrm{Der}^-(X, \mathfrak{g}(\underline{s})) = \sum_{\mu \in T^*} \mathrm{Der}^-(X, \mathfrak{g}(\underline{s}))_\mu$$

decomposes into the direct sum of T-weight spaces. Take $d \in \mathrm{Der}^-(X, \mathfrak{g}(\underline{s}))_\mu$ for some $\mu \ne 0$, and $t \in T$ with $\mu(t) \ne 0$. For arbitrary $u \in X$, we obtain

$$\mu(t)d(u) = (t \cdot d)(u) = [t, d(u)] - d[t, u] = -[d(t), u].$$

Hence $d = \mathrm{ad}_X(-\mu(t)^{-1}d(t)) \in \mathrm{ad}_X\mathfrak{g}(\underline{s})$. According to $t \in T \subset X_0 \cap X_{\bar{0}}$, we have $d \in \mathrm{span}_{\mathbb{F}}\{\mathrm{ad}_X(\partial_i) \mid i \in \mathbf{Y}\}$, thus we only have to determine homogeneous derivations d from X to $\mathfrak{g}(\underline{s})$ of degree $t < 0$ vanishing on given torus T of X_0. For $\mathfrak{g}$ we have $d(X_{-1}) \subset \mathfrak{g}(\underline{s})_{-1+t} = 0$, hence $d(X \cap \ker(\mathrm{ad}\partial_i^p)) \subset X \cap \ker(\mathrm{ad}\partial_i^p)$ and therefore d maps $X \cap \mathfrak{g}(\underline{1})$ into $X \cap \mathfrak{g}(\underline{1})$. Thus d defines by restriction a derivation of $\mathfrak{g}^{(2)}(\underline{1})$. Applying Theorem 2.8 we obtain that $d - \sum_{i \in Y} \alpha_i \mathrm{ad}\partial_i$ vanishes on $\mathfrak{g}^{(2)}(\underline{1})$ for a suitable choice of $\alpha_i \in \mathbb{F}$. Thus we may assume that $\mathfrak{g}^{(2)}(\underline{1}) \subset \ker d$.

Take $\underline{1} \le \underline{k} \le \underline{t}$ to be maximal subject to the condition $\mathfrak{g}^{(2)}(\underline{k}) \subset \ker d$. Then

$$\big[\mathfrak{g}^{(2)}(\underline{1}), d\big(X \cap \overline{HO}(m, m; \underline{k}) \cap \overline{S}(m, m; \underline{k})\big)\big] \subset d(\mathfrak{g}^{(2)}(\underline{k})) = 0,$$

whence $d(X \cap \overline{HO}(m,m;\underline{k}) \cap \overline{S}(m,m;\underline{k})) \subset \big\{D \in \mathfrak{g}(\underline{s}) \big| [\mathfrak{g}^{(2)}(\underline{1}), D] = (0)\big\} = 0$. This is the claim if $\underline{k} = \underline{t}$. Suppose $\underline{k} < \underline{t}$, and let i be an index for which $k_i < t_i$. Take $E \in \mathfrak{g}(\underline{s})$ as $E := \mathrm{T_H}(x^{(p^{k_i}+1)\varepsilon_i})$. Lemma 3.1 proves that $E \notin \ker d$. However, a computation shows that $[E, \mathfrak{g}(\underline{k})_{-1}] \subset X \cap \overline{HO}(m,m;\underline{k}) \cap \overline{S}(m,m;\underline{k}) \subset \ker d$, whence

$$[d(E), \mathfrak{g}(\underline{k})_{-1}] = 0.$$

This means $d(E) \in \sum_{j \in \mathbf{Y}} \mathbb{F}\partial_j$. We may assume that d vanishes on the torus T. Considering eigenvalues we obtain that there exists $\alpha \in \mathbb{F}$ such that

$$d(E) = \alpha\partial_{i'},\ i \in \mathbf{Y_0}.$$

Thus $d' = d - \alpha \mathrm{ad}\partial_i^{p^{k_i}}$ annihilates $\mathfrak{g}^{(2)}(\underline{k}) + \mathbb{F}E$. Lemma 3.1 proves that d' annihilates $\mathfrak{g}^{(2)}(\underline{k} + \varepsilon_i)$. We now proceed by induction.

Thus we may assume that $\mathfrak{g}^{(2)}(\underline{t}) \subset \ker d$. As above we then conclude $d(X) = 0$. □

Remark 3.3. *We use the method for modular Lie algebras [12, Lemmas 5.2.6 and 6.1.3] to prove Lemma 3.1 and Theorem 3.2.*

By virtue of Theorem 3.2, we can determine the negative part of the derivation algebra of $\mathfrak{g}$, $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ as follows:

Proposition 3.4. *We have*

$$\mathrm{Der}^-(X) = \mathrm{span}_{\mathbb{F}}\big(\{(\mathrm{ad}_X(\partial_i))^{p^{k_i}} | i \in \mathbf{Y_0},\ 1 \le k_i < t_i\} \cup \{\mathrm{ad}_X(\partial_i) | i \in \mathbf{Y}\}\big),$$

where $X = \mathfrak{g}$, $\mathfrak{g}^{(1)}$, *or* $\mathfrak{g}^{(2)}$.

Now we only have to investigate the nonnegative part of the derivation algebras. To do that, let us first consider the normalizers of $\mathfrak{g}$, $\mathfrak{g}^{(1)}$, and $\mathfrak{g}^{(2)}$.

Lemma 3.5. $\mathrm{Nor}_W(X) \cap W_t \subseteq (\overline{HO}_t \cap \overline{S}_t)$ $t \in \mathbb{N}$, *where* $X = \mathfrak{g}$, $\mathfrak{g}^{(1)}$, *or* $\mathfrak{g}^{(2)}$.

Proof. Let $X := \mathfrak{g}$, $\mathfrak{g}^{(1)}$, or $\mathfrak{g}^{(2)}$. Suppose

$$E = \sum_{j=1}^{2m} g_j \partial_j \in \mathrm{Nor}_W(X) \cap W_t,$$

where $g_j \in \mathcal{O}(m,m;\underline{t})_{t+1}$, $j \in \mathbf{Y}$. Then there exists $f_i \in \mathcal{O}(m,m;\underline{t})_{t+1}$ satisfying $\Delta(f_i) = 0$, such that

$$[\partial_i, E] = \mathrm{T_H}(f_i),\ i \in \mathbf{Y}.$$

Note that

$$[\partial_i, E] = \sum_{j=1}^{2m} \partial_i(g_j)\partial_j, \qquad \mathrm{T_H}(f_i) = \sum_{j=1}^{2m} (-1)^{\mu(j)\mathrm{p}(f_i)} \partial_j(f_i)\partial_{j'}.$$

We have

$$\partial_i(g_{j'}) = (-1)^{\mu(j)\mathrm{p}(f_i)} \partial_j(f_i), \quad i \in \mathbf{Y}. \tag{3.1}$$

Observe that

$$\begin{aligned}\mathrm{T_H}(\partial_i(f_j)) &= (-1)^{\mu(i)\mathrm{p}(f_j)}[\mathrm{T_H}(f_j), \mathrm{T_H}(x_{i'})] \\ &= (-1)^{\mu(i)\mathrm{p}(f_j)}\big[[\partial_j, E], (-1)^{\mu(i')\mathrm{p}(x_{i'})}\partial_i\big] \\ &= [\partial_i, [\partial_j, E]],\end{aligned}$$

where $i,\ j \in \mathbf{Y}$. Since $[\partial_i, \partial_j] = 0$, we obtain the following equation:

$$\mathrm{T_H}(\partial_i(f_j)) = (-1)^{\mu(i)\mu(j)}\mathrm{T_H}(\partial_j(f_i)), \quad i, j \in \mathbf{Y}. \tag{3.2}$$

Equations (3.1) and (3.2) yield

$$\begin{aligned}&(-1)^{\mu(i)\mathrm{p}(f_j)}\partial_j(g_{i'}) - (-1)^{\mu(i)\mu(j)+\mu(j)\mathrm{p}(f_i)}\partial_i(g_{j'}) \\ =\ & \partial_i(f_j) - (-1)^{\mu(i)\mu(j)}\partial_j(f_i) \in \ker(\mathrm{T_H}) = \mathbb{F}\cdot 1.\end{aligned}$$

Noting that $g_k \in \mathcal{O}(m, m; \underline{t})_{t+1},\ k \in \mathbf{Y}$, we obtain that

$$(-1)^{\mu(i)\mathrm{p}(f_j)}\partial_j(g_{i'}) - (-1)^{\mu(i)\mu(j)+\mu(j)\mathrm{p}(f_i)}\partial_i(g_{j'}) \in \mathbb{F}\cdot 1 \cap \mathcal{O}(m, m; \underline{t})_t.$$

The assumption that $t > 0$ yields

$$(-1)^{\mu(i)\mathrm{p}(f_j)}\partial_j(g_{i'}) = (-1)^{\mu(i)\mu(j)+\mu(j)\mathrm{p}(f_i)}\partial_i(g_{j'}).$$

Since $\mathrm{p}(f_i) = \mu(i) + \mathrm{p}(E) + \overline{1}$, it follows that

$$\partial_i(g_{j'}) = (-1)^{\mu(i)\mu(j)+(\mu(i)+\mu(j))(\mathrm{p}(E)+\overline{1})}\partial_j(g_{i'}).$$

Hence $E \in \overline{HO}$. Since

$$[\partial_i, E] = \mathrm{T_H}(f_i) \in X,\ i \in \mathbf{Y},$$

we obtain $\mathrm{div}[\partial_i, E] = 0$. By virtue of (1.1) we have $E \in \overline{S}$.

Hence $\mathrm{Nor}_W(X) \cap W_t \subseteq (\overline{HO}_t \cap \overline{S}_t)$, $t \in \mathbb{N}$. □

Lemma 3.6. *Let $H = \mathrm{span}_{\mathbb{F}}\{\mathrm{T_H}(x_i x_{i'}) \mid i \in \mathbf{Y_0}\}$, $h_1 = \sum_{i=1}^m x_{i'}\partial_{i'}$. Then $\mathrm{Nor}_W(X) \cap W_0 \subseteq \mathfrak{g}_0 + H + \mathbb{F}\cdot h_1$, where $X = \mathfrak{g}$, $\mathfrak{g}^{(1)}$, or $\mathfrak{g}^{(2)}$.*

Proof. Let $X := \mathfrak{g}$, $\mathfrak{g}^{(1)}$, or $\mathfrak{g}^{(2)}$. Let E be a $\mathbb{Z}_2$-homogeneous element of $\mathrm{Nor}_W(X) \cap W_0$. Then

$$E = \sum_{i=1}^{2m}\sum_{j=1}^{2m}\alpha_{ij}x_i\partial_j, \quad \alpha_{ij} \in \mathbb{F}.$$

Given $i \in \mathbf{Y_0},\ j \in \mathbf{Y_1}$ and $i \neq j'$, we have $\mathrm{T_H}(x_i x_j) \in \mathfrak{g}_0$ and

$$\begin{aligned}[\mathrm{T_H}(x_i x_j), E] &= \Big(\alpha_{i'i'}x_j - \alpha_{j'i'}x_i - \sum_{k=1}^{2m}\alpha_{kj}x_k\Big)\partial_{i'} \\ &+ \Big(\alpha_{i'j'}x_j - \alpha_{j'j'}x_i + \sum_{k=1}^{2m}\alpha_{ki}x_k\Big)\partial_{j'} + \sum_{k\neq i,j}(\alpha_{i'k'}x_j - \alpha_{j'k'}x_i)\partial_{k'}.\end{aligned}$$

Let a_l denote the coefficient of ∂_l in the right side of equation above. Note that $[\mathrm{T_H}(x_i x_j), E] \in \overline{HO} \cap \overline{S}$. Since $\mathrm{p}([\mathrm{T_H}(x_i x_j), E]) = \mathrm{p}(E)$, by virtue of the equality

$$\partial_i(a_{j'}) = (-1)^{\mu(i)\mu(j)+(\mu(i)+\mu(j))(\mathrm{p}(E)+\overline{1})}\partial_j(a_{i'}),$$

an elementary computation shows that

$$(-1)^{\mathrm{p}(E)}\alpha_{ii} + \alpha_{i'i'} = \alpha_{jj} + (-1)^{\mathrm{p}(E)}\alpha_{j'j'}, \quad i \in \mathbf{Y_0},\ j \in \mathbf{Y_1},\ i \neq j'. \tag{3.3}$$

Similarly, by virtue of equations

$$\partial_j(a_{k'}) = (-1)^{\mu(k)\mu(j)+(\mu(k)+\mu(j))(\mathrm{p}(E)+\overline{1})}\partial_k(a_{j'})$$

and

$$\partial_i(a_{k'}) = (-1)^{\mu(k)\mu(i)+(\mu(k)+\mu(i))(\mathrm{p}(E)+\overline{1})}\partial_k(a_{i'}),$$

we obtain that

$$\alpha_{ki} = -(-1)^{\mu(k')\mathrm{p}(E)}\alpha_{i'k'}, \quad i \in \mathbf{Y_0},\ k \in \mathbf{Y}\backslash\{i\} \tag{3.4}$$

and

$$\alpha_{kj} = (-1)^{\mu(k)(\mathrm{p}(E)+\overline{1})}\alpha_{j'k'}, \quad j \in \mathbf{Y_1},\ k \in \mathbf{Y}\backslash\{j\}. \tag{3.5}$$

Case 1: If $\mathrm{p}(E) = \overline{0}$, it is easy to see from (3.3)–(3.5) that

$$\alpha_{ii} + \alpha_{i'i'} = \alpha_{jj} + \alpha_{j'j'}, \quad i \in \mathbf{Y_0},\ j \in \mathbf{Y_1}, \tag{3.6}$$

$$\alpha_{ki} = -\alpha_{i'k'}, \quad i \in \mathbf{Y_0},\ k \in \mathbf{Y}\backslash\{i\}, \tag{3.7}$$

$$\alpha_{kj} = (-1)^{\mu(k)}\alpha_{j'k'}, \quad j \in \mathbf{Y_1},\ k \in \mathbf{Y}\backslash\{j\}. \tag{3.8}$$

Note that $\alpha_{ij} = 0$, whenever $\mu(i) \neq \mu(j)$. We conclude from (3.7) and (3.8) that

$$\alpha_{ij} = (-1)^{\mu(i)+\mu(j')}\alpha_{j'i'}, \quad i, j \in \mathbf{Y},\ i \neq j. \tag{3.9}$$

We may suppose by (3.6) that $\alpha_{ii} + \alpha_{i'i'} = \alpha$, for any $i \in \mathbf{Y_0}$. Applying (3.9) we have

$$E \;=\; \sum_{i=1}^{2m}\alpha_{ii}x_i\partial_i + \frac{1}{2}\sum_{i\neq j}(-1)^{\mu(j')}\alpha_{ij}\mathrm{T_H}(x_i x_{j'}).$$

Moreover,

$$\sum_{i=1}^{2m}\alpha_{ii}x_i\partial_i = -\sum_{i=1}^{m}\alpha_{ii}\mathrm{T_H}(x_i x_{i'}) + \alpha h_1.$$

Hence $E \in \mathfrak{g}_0 + H + \mathbb{F}\cdot h_1$.

Case 2: If $\mathrm{p}(E) = \overline{1}$, then $\alpha_{ij} = 0$ whenever $i, j \in \mathbf{Y}$ and $\mu(i) = \mu(j)$. By virtue of (3.4) and (3.5), we have

$$\alpha_{ki} = (-1)^{\mu(k)}\alpha_{i'k'}, \quad i \in \mathbf{Y_0},\ k \in \mathbf{Y}\backslash\{i\} \tag{3.10}$$

and

$$\alpha_{kj} = \alpha_{j'k'}, \quad j \in \mathbf{Y_1},\ k \in \mathbf{Y}\backslash\{j\}. \tag{3.11}$$

Observe that (3.10) and (3.11) imply that $\alpha_{kl} = (-1)^{\mu(k)}\alpha_{l'k'},\ k,l \in \mathbf{Y}$. Therefore, we obtain that

$$E = \sum_{\mu(i)\neq\mu(j)} \alpha_{ij} x_i \partial_j = \frac{1}{2} \sum_{\mu(i)\neq\mu(j)} (-1)^{\mu(i)} \alpha_{ij} \mathrm{T_H}(x_i x_{j'}).$$

Now we obtain the desired result. □

Note that $HO(m,m;\underline{t})$ is an ideal of $\overline{HO}(m,m;\underline{t})$ and $S'(m,m;\underline{t})$ is an ideal of $\overline{S}(m,m;\underline{t})$. From the definitions we have the following

Proposition 3.7. *X is an ideal of $\overline{HO}(m,m;\underline{t}) \cap \overline{S}(m,m;\underline{t})$, where*

$$X = \mathfrak{g}(m,m;\underline{t}), \mathfrak{g}(m,m;\underline{t})^{(1)},\ \textit{or}\ \mathfrak{g}(m,m;\underline{t})^{(2)}.$$

In conclusion, we can obtain:

Theorem 3.8. $\mathrm{Nor}_W(X) = \overline{HO} \cap \overline{S} \oplus \mathbb{F}\cdot h_1$*, where* $X = \mathfrak{g},\ \mathfrak{g}^{(1)}$*, or* $\mathfrak{g}^{(2)}$.

Proof. Suppose $X := \mathfrak{g},\ \mathfrak{g}^{(1)}$, or $\mathfrak{g}^{(2)}$. Applying Lemmas 3.5 and 3.6 we have $\mathrm{Nor}_W(X) \subset \overline{HO} \cap \overline{S} \oplus \mathbb{F}\cdot h_1$. Note that $[h_1, \mathrm{T_H}(x^\alpha x^u)] = (|u|-1)\mathrm{T_H}(x^\alpha x^u)$, that is $h_1 \in \mathrm{Nor}_W(X)$. Hence by virtue of Proposition 3.7 we obtain $\mathrm{Nor}_W(X) = \overline{HO} \cap \overline{S} \oplus \mathbb{F}\cdot h_1$. □

Remark 3.9. *Let $X = \mathfrak{g},\ \mathfrak{g}^{(1)}$ or $\mathfrak{g}^{(2)}$. Suppose $h := \sum_{i\in\mathbf{Y}} x_i\partial_i$. We obtain* $\mathrm{ad}h$ *is the $\mathbb{Z}$-degree derivation of X, that is for all $\mathbb{Z}$ homogeneous element $A \in X_i$,*

$$[h, A] = iA.$$

Note that $h = -\sum_{i\in\mathbf{Y_0}} \mathrm{T_H}(x_i x_{i'}) + 2h_1$, where $\mathrm{T_H}(x_i x_{i'}) \in \overline{HO} \cap \overline{S}$. We can obtain $\mathrm{Nor}_W(X) = \overline{HO} \cap \overline{S} \oplus \mathbb{F}\cdot h$.

Finally we characterize the derivations of $\mathfrak{g}$, $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$.

Theorem 3.10. *Suppose $X = \mathfrak{g},\ \mathfrak{g}^{(1)}$, or $\mathfrak{g}^{(2)}$, we have*

$$\mathrm{Der}(X) = \mathrm{ad}_X(\overline{HO} \cap \overline{S} \oplus \mathbb{F}\cdot h) \oplus \mathrm{span}_{\mathbb{F}}\{(\mathrm{ad}_X(\partial_i))^{p^{k_i}} | i \in \mathbf{Y_0}, \quad 1 \le k_i < t_i\},$$

where $\mathrm{ad}h$ *is the degree derivation of X. Moreover,*

$$\mathrm{Der}(\mathfrak{g}) \cong \mathrm{Der}(\mathfrak{g}^{(1)}) \cong \mathrm{Der}(\mathfrak{g}^{(2)}).$$

Proof. Consider $\varphi \in \mathrm{Der}_t(X)$ where $t \ge 0$. By virtue of [14, Proposition 2.4] there exists an element $E \in \mathrm{Nor}_W(X) \cap W_t$ such that

$$(\varphi - \mathrm{ad}E)|_X = 0.$$

Then the first part of the assertion holds from Proposition 3.4 and Remark 3.9.
Define

$$\begin{aligned} \varphi_1 : \mathrm{Der}(\mathfrak{g}) &\longrightarrow \mathrm{Der}(\mathfrak{g}^{(1)}) \\ \phi &\longmapsto \phi|_{\mathfrak{g}^{(1)}}, \end{aligned}$$

for $\phi \in \mathrm{Der}(\mathfrak{g})$. Obviously, φ_1 is a monomorphism. Suppose $\phi|_{\mathfrak{g}^{(1)}} = 0$ for any $\phi \in \mathrm{Der}(\mathfrak{g})$. We obtain

$$[\phi(A), (\mathfrak{g}^{(1)})_{-1}] \subset \phi[A, \mathfrak{g}^{(1)}] = 0$$

for any $A \in \mathfrak{g}$. Then $\phi(A) \in (\mathfrak{g}^{(1)})_{-1}$. Note that $[\phi(A), (\mathfrak{g}^{(1)})_0] \subset \phi[A, \mathfrak{g}^{(1)}] = 0$. Hence we have $\phi(A) = 0$ and $\mathrm{Der}(\mathfrak{g}) \cong \mathrm{Der}(\mathfrak{g}^{(1)})$. Similarly, $\mathrm{Der}(\mathfrak{g}^{(1)}) \cong \mathrm{Der}(\mathfrak{g}^{(2)})$. □

4. Outer derivations

Let $X = \mathfrak{g}$, $\mathfrak{g}^{(1)}$, or $\mathfrak{g}^{(2)}$. We denote the outer derivation algebras $\mathrm{Der}_{\mathrm{out}}(X) := \mathrm{Der}(X)/\mathrm{ad}(X)$, which will be determined in this section. Recall $\mathrm{ad}h$ is the $\mathbb{Z}$-degree derivation of X, where $h = \Sigma_{i \in \mathbf{Y_0}} x_i \partial_i$. For future reference, we state the following results.

Lemma 4.1. *The following statements hold in* $\mathrm{Der}\mathcal{O}(m, m; \underline{t})$*:*

(1) $[\partial_i^{p^{k_i}}, \partial_j^{p^{k_j}}] = 0, \quad i, j \in \mathbf{Y_0}, \quad 1 \le k_i < t_i, 1 \le k_j < t_j;$

(2) $[h, \partial_i^{p^{k_i}}] = 0, \quad i \in \mathbf{Y_0}, \quad 1 \le k_i < t_i;$

(3) $[\partial_i^{p^{k_i}}, \overline{HO}] \subset \overline{HO}, \quad i \in \mathbf{Y_0}, \quad 1 \le k_i < t_i;$

(4) $[h, \overline{HO}] \subset \overline{HO}$;

(5) $[\partial_i^{p^{k_i}}, \mathrm{T_H}(a)] = \mathrm{T_H}(\partial_i^{p^{k_i}}(a)), \quad i \in \mathbf{Y_0}, \quad 1 \le k_i < t_i, \quad a \in \mathcal{O}(m, m; \underline{t})$;

(6) $[\partial_i^{p^{k_i}}, \overline{S}] \subset S', \quad i \in \mathbf{Y_0}, \quad 1 \le k_i < t_i;$

(7) $[h, \overline{S}] \subset S'$;

(8) $[h, \mathrm{T_H}(x_j x_{j'})] = 0, \quad j \in \mathbf{Y_0}$.

Proof. (1)–(5) are the direct consequences of [10, Lemma 16], (6)–(8) are obvious. □

Lemma 4.2. *The centralizers of X in W are zero. where $X = \mathfrak{g}$, $\mathfrak{g}^{(1)}$, or $\mathfrak{g}^{(2)}$.*

Proof. Suppose $E = \sum_{i=1}^{2m} a_i \partial_i$ is a centralizer of X in W, where $a_i \in \mathcal{O}(m, m; \underline{t})$. Since

$$[E, \partial_j] = 0, \text{ for all } j \in \mathbf{Y},$$

we have $a_i \in \mathbb{F}$. Since

$$[E, \mathrm{T_H}(x_l x_{l'} - x_k x_{k'})] = 0, \text{ for all } l, k \in \mathbf{Y_0}, l \ne k,$$

we have $a_l = a_{l'} = a_k = a_{k'} = 0$. Hence $E = 0$. □

Now we establish the relationship between $\mathrm{Der}_{\mathrm{out}}(X)$ and the quotient algebra $(\overline{HO} \cap \overline{S})/X$.

Theorem 4.3. *The outer derivation algebras*

$$\mathrm{Der}_{\mathrm{out}}(X) \cong L \oplus (\overline{HO} \cap \overline{S})/X,$$

where

$$\begin{aligned} L &:= \big(\mathbb{F} \cdot h \oplus \mathrm{span}_{\mathbb{F}}\{(\mathrm{ad}_X(\partial_i))^{p^{k_i}} \mid i \in \mathbf{Y_0}, \quad 1 \leq k_i < t_i\} \oplus X\big)\big/X \\ &\cong \mathbb{F} \cdot h \oplus \mathrm{span}_{\mathbb{F}}\{(\mathrm{ad}_X(\partial_i))^{p^{k_i}} \mid i \in \mathbf{Y_0}, \quad 1 \leq k_i < t_i\} \end{aligned}$$

is an Abelian Lie superalgebra. Moreover $(\overline{HO} \cap \overline{S})/X$ *is an ideal of* $\mathrm{Der}_{\mathrm{out}}(X)$*, where* $X = \mathfrak{g}$, $\mathfrak{g}^{(1)}$*, or* $\mathfrak{g}^{(2)}$.

Proof. It is similar to [10, Theorem 18]. □

Remark 4.4. *Suppose* M*,* N_1*,* N_2 *are subspaces of a vector space* V*. If* $N = N_1 \oplus N_2$ *and* $N_1 \subset M$*, then* $M \cap N = N_1 \oplus M \cap N_2$.

Proposition 4.5. $\overline{HO} \cap \overline{S} = \mathfrak{g} \oplus \mathbb{F}\mathrm{T_H}(x_j x_{j'}) \oplus \mathrm{span}_{\mathbb{F}}\big\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \big| i \in \mathbf{Y_0}\big\}$*, for any* $j \in \mathbf{Y_0}$.

Proof. At first, we assert that for any $j \in \mathbf{Y_0}$,

$$\overline{S} = S' \oplus \mathbb{F}\mathrm{T_H}(x_j x_{j'}).$$

Note that

$$\mathrm{div}\big(\mathrm{T_H}(x_j x_{j'})\big) = -2,\ j \in \mathbf{Y_0}.$$

It is sufficient to show that $\overline{S} \subset S' \oplus \mathbb{F}\mathrm{T_H}(x_j x_{j'})$. For any $A \in \overline{S}$, there exists $a \in \mathbb{F}$ such that $\mathrm{div}(A) = a$. Hence $\mathrm{div}(A + \frac{a}{2}\mathrm{T_H}(x_j x_{j'})) = 0$ and $A \in S' \oplus \mathbb{F}\mathrm{T_H}(x_j x_{j'})$.

From [10, Proposition 20], we know that

$$\overline{HO} = HO \oplus \mathrm{span}_{\mathbb{F}}\big\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \mid i \in \mathbf{Y_0}\big\},$$

hence

$$\overline{HO} \cap \overline{S} = HO \cap \overline{S} \oplus \mathrm{span}_{\mathbb{F}}\big\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \mid i \in \mathbf{Y_0}\big\}.$$

Moreover,

$$\overline{HO} \cap \overline{S} = \mathfrak{g} \oplus \mathbb{F}\mathrm{T_H}(x_j x_{j'}) \oplus \mathrm{span}_{\mathbb{F}}\big\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \mid i \in \mathbf{Y_0}\big\}$$

for any $j \in \mathbf{Y_0}$. □

Corollary 4.6. $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}) \cong L \oplus \big(\mathfrak{g} \oplus \mathbb{F}\mathrm{T_H}(x_j x_{j'}) \oplus \mathrm{span}_{\mathbb{F}}\big\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \mid i \in \mathbf{Y_0}\big\}\big)\big/\mathfrak{g}$, *for any* $j \in \mathbf{Y_0}$*, where*

$$\big(\mathfrak{g} \oplus \mathrm{span}_{\mathbb{F}}\big\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \mid i \in \mathbf{Y_0}\big\}\big)\big/\mathfrak{g}$$

is just the odd part of $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g})$ *and*

$$L \oplus \big(\mathbb{F}\mathrm{T_H}(x_j x_{j'}) \oplus \mathfrak{g}\big)\big/\mathfrak{g}$$

is the even part.

Proof. This is a direct consequence of Theorem 4.3 and Proposition 4.5. □

Let A, B be Lie superalgebras. Recall that $A \ltimes_{\varphi} B$ is the semidirect product of A and B with a homomorphism $\varphi : A \to \mathrm{Der}_{\mathbb{F}} B$. Let id denote $a \mapsto a$, $a \in A$ when A is the subalgebra of $\mathrm{Der}_{\mathbb{F}} B$.

Put $\iota := \Sigma_{i=1}^{m} t_i - m$. Let $G := G_{\bar{0}} \oplus G_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{F}$, and $\{h_{-1}, h_0, h_1, \ldots, h_\iota\}$ be an $\mathbb{F}$-basis of $G_{\bar{0}}$, $\{g_1, \ldots, g_m\}$ be an $\mathbb{F}$-basis of $G_{\bar{1}}$. Then G is a $(\Sigma_{i=1}^{m} t_i + 2)$-dimensional Lie superalgebra by means of

$$\begin{aligned}
&[h_{-1}, g_j] = 0,\ j = 1, \ldots, m;\\
&[h_0, g_j] = -2g_j,\ j = 1, \ldots, m;\\
&[h_i, g_j] = 0,\ i = 1, \ldots, \iota,\ j = 1, \ldots, m;\\
&[h_i, h_k] = [g_j, g_l] = 0,\ i, k = -1, \ldots, \iota;\ j, l = 1, \ldots, m.
\end{aligned}$$

Hence we have

$$G = \mathcal{C}(G) \oplus \mathbb{F}h_0 \oplus G_{\bar{1}},$$

where $\mathcal{C}(G)$ is the center of G and

(1) $\mathcal{C}(G) \oplus \mathbb{F}h_0$ is the even part of G;

(2) $G_{\bar{1}}$ is an Abelain subalgebra of G;

Obviously, $\mathrm{ad}(\mathcal{C}(G) \oplus \mathbb{F}h_0)$ is a subalgebra of $\mathrm{Der}(G_{\bar{1}})$. We can obtain

$$G \cong \mathrm{ad}(\mathcal{C}(G) \oplus \mathbb{F}h_0) \ltimes_{\mathrm{id}} G_{\bar{1}}.$$

Theorem 4.7. *The outer derivation algebra* $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g})$ *is isomorphic to the Lie superalgebra* G*.*

Proof. By Corollary 4.6 we have

$$\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}) \cong L \oplus \big(\mathfrak{g} \oplus \mathbb{F}\mathrm{T_H}(x_j x_{j'}) \oplus \mathrm{span}_{\mathbb{F}}\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \mid i \in \mathbf{Y_0}\}\big)/\mathfrak{g},$$

for any $j \in \mathbf{Y_0}$. From Lemma 4.1 we know that both $\big(\mathfrak{g} \oplus \mathrm{span}_{\mathbb{F}}\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \mid i \in \mathbf{Y_0}\}\big)/\mathfrak{g}$ and $L \oplus \big(\mathbb{F}\mathrm{T_H}(x_j x_{j'}) \oplus \mathfrak{g}\big)/\mathfrak{g}$ are Abelian. A direct computation shows that

$$\begin{aligned}
&[\mathrm{T_H}(x_i x_{i'}), x^{(\pi_j \varepsilon_j)}\partial_{j'}] = 0 \quad i, j \in \mathbf{Y_0};\\
&[h, x^{(\pi_j \varepsilon_j)}\partial_{j'}] = -2x^{(\pi_j \varepsilon_j)}\partial_{j'} \quad j \in \mathbf{Y_0};\\
&[\partial_i^{p^{k_i}}, x^{(\pi_j \varepsilon_j)}\partial_{j'}] = \delta_{i=j} x^{((\pi_j - p^{k_j})\varepsilon_j)}\partial_{j'}\\
&\qquad\qquad \in HO \cap S' = \mathfrak{g} \quad i, j \in \mathbf{Y_0}\ 1 \le k_i < t_i.
\end{aligned}$$

Now one can easily establish an isomorphism from $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g})$ to G. □

Now we consider the relationship among $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g})$, $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(1)})$ and $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(2)})$.

Proposition 4.8. *The following statements hold:*

$$\begin{aligned}
&\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(1)}) \cong \mathrm{Der}_{\mathrm{out}}(\mathfrak{g}) \oplus \mathfrak{g}/\mathfrak{g}^{(1)},\\
&\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(2)}) \cong \mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(1)}) \oplus \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)}.
\end{aligned}$$

Proof. Define

$$\begin{array}{rcl}\phi : L \oplus (\overline{HO} \cap \overline{S})/\mathfrak{g}^{(1)} & \longrightarrow & L \oplus (\overline{HO} \cap \overline{S})/\mathfrak{g} \\ A + \mathfrak{g}^{(1)} & \longrightarrow & A + \mathfrak{g}.\end{array}$$

Note that ϕ is a monomorphism, and $\ker(\phi) = \mathfrak{g}/\mathfrak{g}^{(1)}$. By Theorem 4.3 we obtain

$$\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(1)})/\ker(\phi) \cong \mathrm{Der}_{\mathrm{out}}(\mathfrak{g}).$$

Applying Corollary 4.6 we can obtain, for any $j \in \mathbf{Y_0}$,

$$\begin{array}{rcl}\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}) & \cong & L \oplus \big(\mathfrak{g} \oplus \mathbb{F}\mathrm{T_H}(x_j x_{j'}) \oplus \mathrm{span}_{\mathbb{F}}\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \big| i \in \mathbf{Y_0}\}\big)/\mathfrak{g} \\ & \cong & L \oplus \big(\mathfrak{g}^{(1)} \oplus \mathbb{F}\mathrm{T_H}(x_j x_{j'}) \oplus \mathrm{span}_{\mathbb{F}}\{x^{(\pi_i \varepsilon_i)}\partial_{i'} \big| i \in \mathbf{Y_0}\}\big)/\mathfrak{g}^{(1)},\end{array}$$

which is a subalgebra of $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(1)})$ by means of Lemma 4.1. Hence we obtain

$$\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(1)}) \cong \mathrm{Der}_{\mathrm{out}}(\mathfrak{g}) \oplus \mathfrak{g}/\mathfrak{g}^{(1)}.$$

Similarly

$$\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(2)}) \cong \mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(1)}) \oplus \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)}.$$

$\square$

Put $||u|| = |\pi| - (\Sigma_{i' \in u}\pi_i) + |u| - 2$, $u \in \mathbb{B}(m)$, $i \in \mathbf{Y_0}$. Now we can define $G^1 := G \oplus \Lambda(m)$, where $\Lambda(m)$ is the exterior superalgebra. G' has a $\mathbb{Z}_2$-grading structure induced by the $\mathbb{Z}_2$-grading structures of G and $\Lambda(m)$, then G^1 is a $(\Sigma_{i=1}^m t_i + 2 + 2^m)$-dimensional Lie superalgebra by means of

$$\begin{array}{l}[h_{-1}, x^u] = x^u; \\ [h_0, x^u] = ||u||x^u; \\ [h_i, x^u] = 0,\ i = 1, \ldots, \iota; \\ [g_i, x^u] = (-1)^{(i,u)}\delta_{i' \in u} x^{u - \langle i' \rangle},\ i = 1, \ldots, m; \\ [x^u, x^v] = 0,\end{array}$$

for all $x^u, x^v \in \Lambda(m)$, and $(-1)^{(i,u)}$ is determined by the equation $\partial_{i'}(x^u) = (-1)^{(i,u)} x^{u - \langle i' \rangle}$. Obviously, $\mathrm{ad}(G)$ is a subalgebra of $\mathrm{Der}(\Lambda(m))$ and

$$G^1 \cong \mathrm{ad}(G) \ltimes_{\mathrm{id}} \Lambda(m).$$

Recall $\mathfrak{A}_1 = \{\mathrm{T_H}(x^{(\alpha)}x^u) \mid \mathbf{I}(\alpha, u) = \widetilde{\mathbf{I}}(\alpha, u) = \emptyset\}$ is a $\mathbb{Z}_2$-graded subspace of $\mathfrak{g}$ with $\mathfrak{A}_1 = \mathfrak{A}_{1\bar{0}} \oplus \mathfrak{A}_{1\bar{1}}$, where $\mathfrak{A}_{1\bar{\theta}} = \mathfrak{A}_1 \cap \mathfrak{g}_{\bar{\theta}}$, $\bar{\theta} = \bar{0}, \bar{1}$. Notice that $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{A}_1$.

Theorem 4.9. *The outer derivation algebra* $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(1)})$ *is isomorphic to the Lie superalgebra* G^1.

Proof. From Proposition 4.8 we know that $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}^{(1)}) \cong \mathrm{Der}_{\mathrm{out}}(\mathfrak{g}) \oplus \mathfrak{g}/\mathfrak{g}^{(1)}$. Notice that $\mathfrak{g}^{(1)}$ is an ideal of $\mathfrak{g}$. Applying Lemma 4.1 and Theorem 4.7, it is sufficient to consider the operation between $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g})$ and $\mathfrak{A}_1$. By the definition of $\mathfrak{A}_1$ we

know $\mathfrak{A}_1$ is spanned by the elements with the form $\mathrm{T_H}(x^{(\pi-(\Sigma_{r'\in u}\pi_r\varepsilon_r))}x^u)$, a direct computation shows that

$$\begin{aligned}
&\big[\mathrm{T_H}(x_ix_{i'}),\mathrm{T_H}\big(x^{(\pi-(\Sigma_{r'\in u}\pi_r\varepsilon_r))}x^u\big)\big]=\mathrm{T_H}\big(x^{(\pi-(\Sigma_{r'\in u}\pi_r\varepsilon_r))}x^u\big),\ i\in\mathbf{Y_0};\\
&\big[h,\mathrm{T_H}\big(x^{(\pi-(\Sigma_{r'\in u}\pi_r\varepsilon_r))}x^u\big)\big]=||u||\mathrm{T_H}\big(x^{(\pi-(\Sigma_{r'\in u}\pi_r\varepsilon_r))}x^u\big);\\
&\big[\partial_i^{p^{k_i}},\mathrm{T_H}\big(x^{(\pi-(\Sigma_{r'\in u}\pi_r\varepsilon_r))}x^u\big)\big]\subset\mathfrak{g}^{(1)},\ i\in\mathbf{Y_0},\ 1\le k_i<t_i;\\
&\big[x^{(\pi_i\varepsilon_i)}\partial_{i'},\mathrm{T_H}\big(x^{(\pi-(\Sigma_{r'\in u}\pi_r\varepsilon_r))}x^u\big)\big]=\delta_{i'\in u}\mathrm{T_H}\big(x^{(\pi-(\Sigma_{r'\in u}\pi_r\varepsilon_r)+\pi_i\varepsilon_i)}\partial_{i'}x^u\big),\ i\in\mathbf{Y_0}.
\end{aligned}$$

Note that $[\mathfrak{A}_1,\mathfrak{A}_1]\subset\mathfrak{g}^{(1)}$. Hence we can easily establish an isomorphism from $\mathrm{Der_{out}}(\mathfrak{g}^{(1)})$ to G^1. □

Put $G^2:=G^1\oplus\mathbb{F}f$. Define the even part of G^2 as follows:

$$\begin{aligned}
&G^2_{\bar 0}=G^1_{\bar 0}\oplus\mathbb{F}f \quad \text{if } m \text{ is even},\\
&G^2_{\bar 0}=G^1_{\bar 0} \qquad\quad\ \text{if } m \text{ is odd}.
\end{aligned}$$

Then G^2 is a $(\Sigma_{i=1}^m t_i+3+2^m)$-dimensional Lie superalgebra by means of

$$\begin{aligned}
&[h_{-1},f]=2f;\\
&[h_0,f]=-4f;\\
&[h_i,f]=0,\ i=1,\dots,\iota;\\
&[g_i,f]=0,\ i=1,\dots,m;\\
&[x^u,f]=0,\ x^u\in\Lambda(m).
\end{aligned}$$

Moreover, $G^2\cong\mathrm{ad}(G^1)\ltimes_{\mathrm{id}}\mathbb{F}f$.

Recall $\mathfrak{g}^{(1)}=\mathfrak{g}^{(2)}\oplus(\mathfrak{g}^{(1)})_{\xi-4}$ and

$$(\mathfrak{g}^{(1)})_{\xi-4}=\mathrm{span}_{\mathbb{F}}\left\{\mathrm{T_H}\left(x^{(\pi-\varepsilon_i)}x^{\omega-\langle i'\rangle}-\sum_{r\in\mathbf{Y_0}\setminus\{i\}}\Gamma_r^i\big(x^{(\pi-\varepsilon_i)}x^{\omega-\langle i'\rangle}\big)\right)\,\middle|\, i\in\mathbf{Y_0}\right\}.$$

Notice that $\dim(\mathfrak{g}^{(1)})_{\xi-4}=1$.

Theorem 4.10. *The outer derivation algebra* $\mathrm{Der_{out}}(\mathfrak{g}^{(2)})$ *is isomorphic to the Lie superalgebra* G^2.

Proof. Similar to Theorem 4.9, it is sufficient to consider the operation between $\mathrm{Der_{out}}(\mathfrak{g}^{(1)})$ and $(\mathfrak{g}^{(1)})_{\xi-4}$. For any

$$M=a\mathrm{T_H}\left(x^{(\pi-\varepsilon_i)}x^{\omega-\langle i'\rangle}-\sum_{r\in\mathbf{Y_0}\setminus\{i\}}\Gamma_r^i\big(x^{(\pi-\varepsilon_i)}x^{\omega-\langle i'\rangle}\big)\right)\in(\mathfrak{g}^{(1)})_{\xi-4},$$

where $a\in\mathbb{F}$, we can obtain:

$$\begin{aligned}
&[\mathrm{T_H}(x_ix_{i'}),M]=2M,\ i\in\mathbf{Y_0};\\
&[h,M]=-4M;\\
&[\partial_j^{p^{k_j}},M]=\mathrm{T_H}\left(x^{(\pi-p^{k_j}\varepsilon_j-\varepsilon_i)}x^{\omega-\langle i'\rangle}-\sum_{r\in\mathbf{Y_0}\setminus\{i\}}\Gamma_r^i\big(x^{(\pi-p^{k_j}\varepsilon_j-\varepsilon_i)}x^{\omega-\langle i'\rangle}\big)\right)\\
&\in\mathfrak{g}^{(2)},\ j\in\mathbf{Y_0};\\
&[x^{(\pi_i\varepsilon_i)}\partial_{i'},M]=0,\ i\in\mathbf{Y_0}.
\end{aligned}$$

Note that $[\mathfrak{A}_1,(\mathfrak{g}^{(1)})_{\xi-4}]=0$. Hence we can easily establish an isomorphism from $\mathrm{Der_{out}}(\mathfrak{g}^{(2)})$ to G^2. □

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